

LS category, foliated spaces and
transverse invariant measure
(*Categoría LS , espacios foliados y
medida transversa invariante*)

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Abstract

The LS category is a homotopy invariant of topological spaces introduced by Lusternik and Schnirelmann in 1934, which was originally motivated by problems of variational calculus. It is defined as the minimum number of contractible open subsets needed to cover a space. Besides its original variational application, it became an important tool in homotopy theory, and it was applied in other different areas like robotics.

Many variants of the LS category has been given; in particular, E. Macías and H. Colman introduced a tangential version for foliations, where they used leafwise contractions to transversals. In this thesis, the following new versions of the tangential LS category are introduced:

Measurable category: It is the direct adaptation of the tangential LS category to measurable laminations, where we have an ambient measure space with a leaf topology (the ambient space topology is missing). Then it is natural to define a measurable category as the minimum number of measurable (leafwise) open sets contractible to transversals by measurable (leafwise) continuous deformations. This version is relevant because, by using the obvious forgetful functor from laminations to measurable laminations, it provides a lower bound of the tangential LS category, which is easier to deal with (it is easier to handle measurable functions than continuous ones).

Measurable Λ -category: This version is also defined for measurable laminations, but now a transverse invariant measure Λ is used to “count” the tangential deformations of measurable open sets to transversals; i.e., it is the infimum of the sums of the Λ -measures of the transversals resulting from tangential deformations of the sets of a measurable open covering.

Topological Λ -category: This version is defined for laminations; i.e., there is also an ambient space topology. Then its definition is like the measurable Λ -category, depending on a transverse invariant measure Λ , but now we take coverings that are open on the ambient space, as well as tangential deformations that are continuous on the ambient space.

Secondary Λ -category: The computations in examples show that, unfortunately, the topological Λ -category vanishes very easily (e.g. when the leaves are dense). For that reason, when the topological Λ -category vanishes, an induced secondary invariant is defined as

the rate of convergence to zero of the expression giving the Λ -category when the “length” of the tangential deformations increases to infinity.

Dynamical category: It is defined like the topological Λ -category, but, instead of using a transverse invariant measure, the diameter given by an ambient metric is used to “measure” the size of the transversals resulting from tangential deformations. The main advantage of this definition is that it does not require any transverse invariant measure, whose existence is a very restrictive condition. In the case of smooth foliations on closed Riemannian manifolds, the positivity or nullity of the dynamical category depends only on the foliation.

Secondary dynamical category: It is the secondary invariant associated to the dynamical category in the same way as the secondary Λ -category is associated to the topological Λ -category.

The main achievements of the thesis are:

- preliminary studies of transverse invariant measures and measurable cohomology,
- computations of these variations of the tangential category in examples,
- corresponding versions of the main theorems about the classical LS category, and
- new theorems that are special of the lamination setting.

More precisely, the following kind of results are given:

Extension of transverse invariant measures: It is proved that any transverse invariant measure extends to a measure on the ambient space, which is unique if some condition of coherency is assumed. This extension can be considered as a pairing of the given transverse invariant measure with the counting measure on the leaves.

Measurable cohomology: This is a new version of cohomology introduced for measurable laminations. The definition involves tangential cochains which give measurable functions when applied to transversely measurable families of simplices. Some standard properties of cohomology are extended to this setting, and it is computed in some examples.

Examples: Expressions of the above versions of the tangential category are given in the cases of compact leaves, dense leaves or suspensions with Rohlin groups.

Homotopy invariance: It is proved that these versions of the tangential LS category are invariant by tangential homotopy equivalences compatible with the structure involved in their definitions (transverse invariant measures or Lipschitz types of metrics).

Transverse invariance: The positivity or nullity of the above primary categories, and the corresponding secondary categories as well, are shown to depend only on the transverse holonomy pseudogroup. For this purpose, corresponding versions of primary and

secondary LS categories are defined for pseudogroups, and they turn out to be invariant by pseudogroup equivalences.

Growth of pseudogroups: We prove that the secondary categories of a pseudogroup are related with its growth.

Dimensional upper bound: The dimension of a manifold is an upper bound of its LS category. This was generalized to the tangential category by Singhof-Vogt. We also prove corresponding versions for our primary categories.

Cohomological lower bound: The usual lower bound of the LS category by the cup length is extended to our new versions of the tangential LS category. The secondary tangential categories require corresponding secondary versions of the cup length.

Semicontinuity: Singhof-Vogt have proved that the tangential LS category is upper semicontinuous on the space of foliations on a fixed manifold. By using the same methods, we have also proved upper semicontinuity for our primary tangential categories, and certain lower semi-continuity for the secondary ones.

Critical points: For smooth functions on manifolds, one of the first theorems on the classical LS category states that it is a lower bound of the number of critical points, establishing its connection with variational calculus. That kind of result was missing even for the original tangential LS category. A version of that relation with critical points was proved for all versions of the tangential categories, including the original one of Colman-Macías. This could be considered as the most important result of the thesis. We use critical sets instead of critical points, which are “measured” with the tools used in the tangential categories (counting, measures or diameters). Secondary approaches to the critical sets are also considered in the case of secondary tangential categories. For the case of smooth functions on manifolds, the use of critical sets improves the classical result, showing the relevance of this point of view. For these kind of theorems, we have used Hilbert laminations with the purpose of applying them to “foliation variational calculus” in the future.

Introduction

The LS category is a homotopy invariant given by the minimum of open subsets, contractible within a topological space, needed to cover it. It was introduced by Lusternik and Schnirelmann in 1934 in the setting of variational calculus [28]. Many variants of this notion has been given. In particular, E. Macías and H. Colman introduced a tangential version for foliations, where they used leafwise contractions to transversals [7, 8].

In this work, we introduce and study several new versions of the tangential LS category. The first two of them, called measurable category and measurable Λ -category, are defined for measurable laminations [5, 4], which are laminations where the ambient topology is removed and only the leaf topology and ambient measurable structure remain; thus we use tangential deformations of open sets to transversals that are leafwise continuous and measurable on the ambient space. Another version, called topological Λ -category, is defined for (topological) laminations. It involves tangential deformations of open sets to transversals that are continuous on the ambient space. In the measurable and topological Λ -categories, a transverse invariant measure Λ is used to “count” the number of those open sets by measuring the transversals resulting from their leafwise deformations. Unfortunately, the measurable and topological Λ -categories vanish very easily. However, in that case, a secondary (topological) Λ -category is defined giving the rate of convergence to zero of the expression involved in the definition of the topological Λ -category. This new type of invariant provides additional information of the lamination.

Finally, we pick up an ambient metric and, instead of a transverse invariant measure, we use the diameter to measure the transversals resulting from the tangential deformations of open sets. In this way, we introduce another version of tangential category, called the dynamical category. It turns out that, when the ambient space is compact, the positivity or nullity of the dynamical category is independent of the chosen metric, obtaining an invariant of the lamination. The dynamical category also vanishes very easily, and, in this case, a secondary dynamical category can be defined as before.

These new versions of the tangential LS category are computed for many examples. Also, many well known results of the classical LS category are adapted to these versions: homotopy invariance, cohomological lower bound (with the cup length), semicontinuity, dimensional upper bound and relation with critical points. Our study of the relation with critical points also gives an inequality for the tangential LS category of Colman-Macías, and it even improves the known inequality for the classical LS category. Moreover new types of results are proved, like transverse invariance, and relation with

growth. It looks very likely that there exists a relation between the secondary dynamical category and the entropy, even though we still did not prove any result in that direction.

This thesis is divided into four parts. The first one of them uses measurable laminations. We define their measurable category and measurable Λ -category and adapt some of the classical results of the LS category, like the cohomological lower bound by the cup length, the dimensional upper bound, or the lower bound for the number of critical points of a smooth function. First of all, in measurable laminations, the topological or differentiable properties always refer to the leaf topology, and the measurable terms refer to the measurable ambient structure. For example, a measurable open set is an open set in the leaf topology that is measurable for the ambient structure. The measurable category, a direct adaptation of the tangential LS category, is the minimum number of tangentially categorical measurable open sets covering the space. This adaptation is interesting since it gives a lower bound for the usual tangential category. Given a transverse invariant measure Λ , the measurable Λ -category is the infimum of the sums of the measures of transversals resulting from measurable and continuous deformations of measurable open sets covering the space. We introduce the theory of measurable singular cohomology for measurable laminations and prove that the corresponding cup length is a lower bound for the measurable category. We shall prove that the measurable category is a lower bound for the number of critical sets of a differentiable function, where the critical sets are defined by using the leafwise gradient flow of the function. The measurable Λ -category is a lower bound for the Λ -measure of the critical set.

In the second part, we discuss a topological version of the measurable Λ -category for laminations. It is defined in a similar way by taking usual tangential deformations to transversals of categorical open sets (now the ambient topology is considered). In a usual lamination, the topological Λ -category is greater than the measurable Λ -category. We obtain that its nullity or positivity is a transverse invariant (not only homotopy invariant). We achieve a similar dimensional upper bound given by the product of the dimension of the foliation and the measure of any complete transversal. This invariant also is an upper semicontinuous map in the space of foliations with transverse invariant measures on a compact manifold, adapting a similar result of tangential category [33]. For functions with closed critical sets, we show that the usual tangential category is a lower bound for the number of critical sets, and the Λ -category is a lower bound for the Λ -measure of the leafwise critical points.

In the third part, we introduce a secondary version of the topological Λ -category for laminations on compact spaces. When the Λ -category is zero, we define the (n, Λ) -category in the same way by taking only deformations with length $\leq n$, where the length refers to the minimum length of chains of plaques of a foliated atlas covering the leafwise paths of the deformation. The secondary category is defined as the growth type of the sequence of inverses of (n, Λ) -categories. We study its relation with the growth of groups and pseudogroups [36, 16], obtaining that it is in fact a transverse invariant.

We explore the meaning of the classical results from this exotic point of view, like the cohomological bound, semicontinuity or the relation with the set of leafwise critical set of a smooth function.

In the final part, we introduce the dynamical category and secondary dynamical category, and we show that they satisfy corresponding versions of the above properties, like the cohomological bound, the dimensional bound or the semicontinuity. Also, it is a lower bound for the sum of the diameters of the leafwise critical sets of a smooth function and the transverse invariance.

We point out that another possibility of generalization could be given by using harmonic measures, but this point of view is left for further studies.

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Part 1

Measurable category

CHAPTER 1

Measurable laminations

1. MT-spaces and measurable laminations

A *measurable topological space*, or *MT-space*, is a set X equipped with a σ -algebra and a topology. Usually, measure theoretic concepts will refer to the σ -algebra of X , and topological concepts will refer to its topology; in general, the σ -algebra is different from the Borel σ -algebra induced by the topology. An *MT-map* between MT-spaces is a measurable continuous map. An *MT-isomorphism* is a map between MT-spaces that is a measurable isomorphism and a homeomorphism.

Trivial examples are topological spaces with de Borel σ -algebra, and measurable spaces with the discrete topology. Let X and Y be MT-spaces. Suppose that there exists a measurable embedding $i : X \rightarrow Y$ that maps measurable sets to measurable sets. Then X is called an *MT-subspace* of Y . Notice that if X and Y are standard, the measurability of i means that it maps Borel sets to Borel sets [34]. The product $X \times Y$ is an MT-space too with the product topology and the σ -algebra generated by products of measurable sets of X and Y .

Let R be an equivalence relation on an MT-space X . In order to give an MT-structure to the quotient X/R , consider the quotient topology and the σ -algebra generated by the projections of measurable saturated sets of X .

A *Polish space* is a completely metrizable and separable topological space. A *standard Borel space* is a measurable space isomorphic to a Borel subset of a Polish space. Let T be a standard Borel space and let P be a Polish space. $P \times T$ will be endowed with the structure of MT-space defined by the σ -algebra generated by products of Borel subsets of T and Borel subsets of P , and the product of the discrete topology on T and the topology of P .

A *measurable chart* on an MT-space X is an MT-isomorphism $\varphi : U \rightarrow P \times T$, where U is open and measurable in X , T is a standard Borel space, and P is locally compact, connected and locally path connected Polish space; let us remark that P and T depend on the chart. The sets $\varphi^{-1}(P \times \{*\})$ are called *plaques* of φ , and the sets $\varphi^{-1}(\{*\} \times T)$ are called *transversals* associated to φ . A *measurable atlas* on X is a countable family of measurable charts whose domains cover X . A *measurable lamination* is an MT-space that admits a measurable atlas. Observe that we always consider countable atlases, therefore the ambient space is also a standard space. The connected components of X are called its *leaves*. An example of measurable lamination is a usual foliation with its Borel σ -algebra and the leaf topology. According to this definition, the leaves are second countable connected manifolds, but they may not be Hausdorff. If $P \simeq \mathbb{R}^m$, it is possible to define a concept of

C^r tangential structure; in this setting, it cannot be defined as a maximal atlas with (tangentially) C^r changes of coordinates because the atlases are required to be countable, but we proceed as follows. A measurable atlas is said to be (tangentially) C^r if its coordinate changes are (tangentially) C^r . Then a C^r structure is an equivalence class of C^r measurable atlases, where two C^r measurable atlases are equivalent if their union is a C^r measurable atlas.

The term “lamination” (or “measurable lamination”) is commonly used when the leaves are manifolds. Thus the term “measurable Polish lamination” could be more appropriate in our setting. But we simply write “measurable lamination” for the sake of simplicity. We remark also the fact that the tangential model may be a separable Hilbert space and we can speak also about *measurable Hilbert laminations*.

A measurable subset $T \subset X$ is called a *transversal* if its intersection with each leaf is countable [18]; these are slightly more general than the transversals of [4]. Let $\mathcal{T}(X)$ be the family of transversals of X . This set is closed under countable unions and intersections, but it is not a σ -algebra. A transversal meeting all leaves is called *complete*.

A *measurable holonomy transformation* is a measurable isomorphism $\gamma : T \rightarrow T'$, for $T, T' \in \mathcal{T}(X)$, which maps each point to a point in the same leaf. A *transverse invariant measure* on X is a σ -additive map, $\Lambda : \mathcal{T}(X) \rightarrow [0, \infty]$, invariant by measurable holonomy transformations. The classical definition of transverse invariant measure in the context of foliated spaces is a measure on topological transversals invariant by holonomy transformations (see e.g. [6]). These two notions of transverse invariant measures agree for foliated spaces [9].

Our principal tools in this setting are the following two results.

PROPOSITION 1.1 (Lusin, see e.g. [34]). *Let X and Y be standard Borel spaces and $f : X \rightarrow Y$ a measurable map with countable fibers. Then $f(X)$ is Borel in Y and there exists a measurable section $s : f(X) \rightarrow X$ of f . In particular, if f is injective, then s is a Borel isomorphism. Moreover there exists a countable Borel partition, $X = \bigcup_i X_i$, so that each restriction $f|_{X_i}$ is injective.*

THEOREM 1.2 (Kunugui, Novikov, see e.g. [34]). *Let $\{V_n\}_{n \in \mathbb{N}}$ be a countable base for a Polish space P . Let $B \subset P \times T$ be a Borel set such that $B \cap (P \times \{t\})$ is open for every $t \in T$. Then there exists a sequence $\{B_n\}_{n \in \mathbb{N}}$ of Borel sets of T such that*

$$B = \bigcup_n (V_n \times B_n) .$$

LEMMA 1.3. *Let $\varphi_i : U_i \rightarrow P \times T_i$ and $\varphi_j : U_j \rightarrow P \times T_j$ be measurable charts of X . There exists a sequence of Borel sets of T_i , $\{B_n\}_{n \in \mathbb{N}}$, and a base of P , $\{V_n\}_{n \in \mathbb{N}}$, such that $\varphi_i(U_i \cap U_j) = \bigcup_n (V_n \times B_n)$ and $\varphi_j \circ \varphi_i^{-1}(x, t) = (g_{ijn}(x, t), f_{ijn}(t))$ for $(x, t) \in V_n \times B_n$, where each f_{ijn} is a Borel isomorphism and each g_{ijn} is an MT-map.*

PROOF. We apply Theorem 1.2 to $\varphi_j(U_i \cap U_j)$, with a base $\{V_n\}_{n \in \mathbb{N}}$ consisting of connected open sets of P , and obtain a family of sets $V_n \times B'_n$

such that $\varphi_j(U_i \cap U_j) = \bigcup_n (V_n \times B'_n)$. Now we apply Theorem 1.2 to each set $\varphi_i \circ \varphi_j^{-1}(V_k \times B'_k)$, $k \in \mathbb{N}$. We obtain sequences $B'_{k,n}$ such that

$$\varphi_i \circ \varphi_j^{-1}(V_k \times B'_k) = \bigcup_n (V_n \times B'_{k,n}), \quad k \in \mathbb{N}.$$

The sets $\varphi_j \circ \varphi_i^{-1}(V_n \times \{t\})$, $t \in B'_{k,n}$, are contained in a single plaque of the form $\mathbb{R}^n \times \{*\}$ since each V_n is connected. Hence

$$\varphi_j \circ \varphi_i^{-1}(x, t) = (g_{ijkn}(x, t), f_{ijkn}(t))$$

for $(x, t) \in V_n \times B'_{k,n}$. We shall show that f_{ijkn} is bijective. If there exist $t, t' \in T_i$ with $f_{ijkn}(t) = f_{ijkn}(t') = t''$, then $g_{ijkn}(V_n \times \{t\})$ and $g_{ijkn}(V_n \times \{t'\})$ are contained in the plaque $V_k \times \{t''\}$ of $V_k \times B'_k$, but this plaque is image by $\varphi_j \circ \varphi_i^{-1}$ of a given connected open set since φ_j and φ_i are homeomorphisms, and V_k is connected. Thus the image by $\varphi_i \circ \varphi_j^{-1}$ of the plaque $V_k \times \{t''\}$ is contained in a single plaque of U_i . This contradicts $t \neq t'$.

It is easy to show that the maps f_{ijkn} are measurable since they can be given as a composition of a projection with the restriction of the cocycle map $\varphi_j \circ \varphi_i^{-1}$ to $B'_{k,n}$. Finally, the maps f_{ijkn} are Borel isomorphisms to their images by Proposition 1.1. \square

DEFINITION 1.4. A measurable foliated atlas \mathcal{U} is called *regular* if, for each chart $(U, \varphi) \in \mathcal{U}$, there exists another measurable foliated chart (W, ψ) such that the closure of each plaque in U is compact, $\overline{U} \subset W$, $\varphi = \psi|_U$, and, for every pair of charts $(U_1, \varphi_1), (U_2, \varphi_2) \in \mathcal{U}$, each plaque of (U_1, φ_1) meets at most one plaque of (U_2, φ_2) . The set \overline{U} is also measurable for all $(U, \varphi) \in \mathcal{U}$.

This definition is weaker than the corresponding one for usual foliations (see *e.g.* [6]). The locally finite condition cannot be considered in the measurable setting since there is no ambient topology. The following result follows from Lemma 1.3. Also, in measurable Hilbert laminations, we cannot assume that plaques would be relative compact but we can change this condition by requiring to be bounded relative to some measurable leafwise metric (a metric on the leaves that varies mesurably on the ambient space).

COROLLARY 1.5. *A measurable lamination with a measurable foliated atlas, such that each chart meets a finite number of charts, admits a regular measurable foliated atlas.*

From now on, we always consider measurable laminations that admit regular measurable foliated atlases.

2. Examples

EXAMPLE 1.6 (Usual laminations and foliations). Usual topological laminations induce measurable laminations when we consider the leaf topology and the ambient measurable structure. In fact we can define a functor from the category of topological laminations and foliated maps (those that map leaves to leaves) to the category of measurable laminations and MT-maps.

EXAMPLE 1.7 (Standard Borel spaces). Standard Borel spaces are MT-maps when we consider the discrete topology. This kind of MT-spaces are our model of measurable transversals in the setting of measurable laminations.

EXAMPLE 1.8 (Measurable wedge). Let \mathcal{F} and \mathcal{G} be measurable laminations, and let T and T' be complete transversals consisting of isolated points in leaves. Suppose that there exists a measurable bijection $\gamma : T \rightarrow T'$ such that $\gamma : T/\mathcal{F} \rightarrow T'/\mathcal{G}$ is bijective (the bijection take points of each leaf to a single leaf in both directions). The *measurable wedge* $\mathcal{F} \vee_{\gamma}^T \mathcal{G}$, relative to the pair (T, γ) , is the quotient MT-space of $\mathcal{F} \sqcup \mathcal{G}$ by the relation $t \sim \gamma(t)$ for $t \in T$. The measurable wedge is a measurable lamination since T and $T' = \gamma(T)$ consist of isolated points on the leaves. If T (and T') meets each leaf in only one point, then the measurable wedge is a measurable lamination where any leaf is a wedge of two leaves by the latest condition on γ . Unfortunately, in many measurable laminations there is no measurable transversal with this property.

EXAMPLE 1.9 (Measurable suspensions). Let P be a connected, locally path connected and semi-locally 1-connected Polish space, and let S be a standard space. Let $\text{Meas}(S)$ denote the group of measurable transformations of S . Let

$$h : \pi_1(P, x_0) \rightarrow \text{Meas}(S)$$

be a homomorphism. Let \tilde{P} the universal covering of P and consider the action of $\pi_1(P, x_0)$ on the MT-space $\tilde{P} \times S$ given by

$$g \cdot (x, y) = (xg^{-1}, h(g)(y)) .$$

The corresponding quotient MT-space, $\tilde{P} \times_h S$, will be called the *measurable suspension* of h . $\tilde{P} \times_h S$ is a measurable lamination, $\{*\} \times S$ is a complete transversal, and its leaves are covering spaces of P .

EXAMPLE 1.10 (Measurable graphs). Measurable graphs are measurable laminations such that every leaf is a graph in the classical sense. In this setting, any plaque is a finite wedge of open intervals. Of course, the measurable wedge of measurable graphs is a measurable graph.

CHAPTER 2

Coherent extension of an invariant measure

Transverse invariant measures of foliated spaces play an important role in the study of their transverse dynamics. They are measures on transversals invariant by holonomy transformations. There are many interpretations of transverse invariant measures; in particular, they can be extended to generalized transversals, which are defined as Borel sets that meet each leaf in a countable set [9]. Here, we show that indeed invariant measures can be extended to the σ -algebra of all Borel sets becoming an “ambient” measure (a measure on the ambient space) and this extension is unique with suitable conditions [24].

1. Case of a product measurable lamination

In this section, we take measurable laminations of the form $P \times T$, where T is a standard measurable space and P a connected and locally connected Polish space. We assume that a new topology is given in this space as follows. All standard Borel spaces are Borel isomorphic to a finite set, \mathbb{Z} or the interval $[0, 1]$ (see [34, 35]). Identify $P \times T$ with $P \times \mathbb{Z}$, $P \times [0, 1]$ or $P \times A$ (A finite), via a Borel isomorphism. We work with two topologies in $T \times P$. On the one hand, the topology of the MT-structure is the product of discrete topology on T and the topology of P . On the other hand, the topology is the product of the topology of $[0, 1]$, \mathbb{N} or P with the topology of P ; the term “open set” is used with this topology. The σ -algebra of the MT-structure on $P \times T$ is generated by these “open sets”. Let $\pi : P \times T \rightarrow T$ be the first factor projection.

PROPOSITION 2.1 (R.Kallman [22]). *If $B \subset P \times T$ is a Borel set such that $B \cap (P \times \{t\})$ is σ -compact for all $t \in T$, then $\pi(B)$ is a Borel set. Moreover there exists a Borel subset $B' \subset B$ such that $\#(B' \cap (P \times \{t\})) = 1$ if $B \cap (P \times \{t\}) \neq \emptyset$, and $\#(B' \cap (P \times \{t\})) = 0$ otherwise.*

For any measurable space $(X, \mathcal{M}, \Lambda)$, the *completion* of \mathcal{M} with respect to Λ is the σ -algebra

$$\mathcal{M}_\Lambda = \{ Z \subset X \mid \exists A, B \in \mathcal{M}, A \subset Z \subset B, \Lambda(B \setminus A) = 0 \}.$$

The measure Λ extends in a natural way to \mathcal{M}_Λ by defining $\Lambda(Z) = \Lambda(A) = \Lambda(B)$ for Z, A, B as above.

Now, let Λ be a Borel measure on T . Define

$$\pi(\mathcal{B}^*, \mathcal{B}_\Lambda) = \{ B \subset \mathcal{B}^* \mid \pi(B \cap U) \in \mathcal{B}_\Lambda \ \forall \text{ open } U \subset T \times P \},$$

where \mathcal{B} and \mathcal{B}^* are the Borel σ -algebras of T and $P \times T$, respectively.

PROPOSITION 2.2. *$\pi(\mathcal{B}^*, \mathcal{B}_\Lambda)$ is closed under countable unions.*

PROOF. This is obvious since, for any countable family $\{B_n\} \subset \mathcal{B}^*$, we obtain $\bigcup_n B_n \in \mathcal{B}^*$ and

$$\pi \left(\left(\bigcup_n B_n \right) \cap U \right) = \bigcup_n \pi(B_n \cap U) \in \mathcal{B}_\Lambda$$

for any open subset $U \subset P \times T$. \square

REMARK 1. If Λ is σ -finite (*i.e.*, T is a countable union of Borel sets with finite Λ -measure), then $\pi(\mathcal{B}^*, \mathcal{B}_\Lambda) = \mathcal{B}^*$: by Exercise 14.6 in [23], any set in \mathcal{B}^* projects onto an analytic set, which is Λ -measurable since Λ is σ -finite [34, Theorem 4.3.1].

REMARK 2. If $B \in \pi(\mathcal{B}^*, \mathcal{B}_\Lambda)$ and U is an open set, then $B \cap U \in \pi(\mathcal{B}^*, \mathcal{B}_\Lambda)$. By Proposition 2.1, $\pi(\mathcal{B}^*, \mathcal{B}_\Lambda)$ contains the Borel sets with σ -compact intersection with the plaques $P \times \{t\}$.

Now, we want to extend Λ to all Borel sets satisfying the conditions of a measure. Let \mathcal{B}^{**} denote the Borel σ -algebra of $P \times T \times P \times T$, $\tilde{\pi}$ the natural projection $P \times T \times P \times T \rightarrow T \times T$, and $\langle \mathcal{B}_\Lambda \times \mathcal{B}_\Lambda \rangle$ the σ -algebra generated by sets of the form $A \times B$ for $A, B \in \mathcal{B}_\Lambda$.

LEMMA 2.3. *If $B, B' \in \pi(\mathcal{B}^*, \mathcal{B}_\Lambda)$, then $B \times B' \in \tilde{\pi}(\mathcal{B}^{**}, \langle \mathcal{B}_\Lambda \times \mathcal{B}_\Lambda \rangle)$.*

PROOF. Since $B, B' \in \mathcal{B}^*$, we have $B \times B' \in \mathcal{B}^{**}$. Observe that every open set $U \subset P \times T$ is a countable union of products of open sets. Write $U = \bigcup_{n=1}^\infty (U_n \times V_n)$ with U_n and V_n open subsets of T and P , respectively. Then

$$\begin{aligned} \tilde{\pi}((B \times B') \cap U) &= \tilde{\pi} \left((B \times B') \cap \bigcup_{n=1}^\infty (U_n \times V_n) \right) \\ &= \tilde{\pi} \left(\bigcup_{n=1}^\infty ((B \cap U_n) \times (B' \cap V_n)) \right) = \bigcup_{n=1}^\infty \tilde{\pi}((B \cap U_n) \times (B' \cap V_n)) \\ &= \bigcup_{n=1}^\infty (\pi(B \cap U_n) \times \pi(B' \cap V_n)), \end{aligned}$$

which is in $\langle \mathcal{B}_\Lambda \times \mathcal{B}_\Lambda \rangle$. \square

DEFINITION 2.4. For $B \in \pi(\mathcal{B}^*, \mathcal{B}_\Lambda)$, let

$$\tilde{\Lambda}(B) = \int_T \#(B \cap (P \times \{t\})) d\Lambda(t) = \int_T \left(\int_{P \times \{t\}} \chi_{B \cap (\{t\} \times P)} d\nu \right) d\Lambda(t),$$

where ν denotes the counting measure and χ_X the characteristic function of a subset $X \subset \{t\} \times P$.

REMARK 3. A measure on T induces a transverse invariant measure on $P \times T$. When B is a generalized transversal, $\tilde{\Lambda}(B)$ is the value of this transverse invariant measure on B . Therefore Definition 2.4 defines an extension of this transverse invariant measure to a map $\tilde{\Lambda} : \pi(\mathcal{B}^*, \mathcal{B}_\Lambda) \rightarrow [0, \infty]$.

PROPOSITION 2.5. *On $B \in \pi(\mathcal{B}^*, \mathcal{B}_\Lambda)$, $\tilde{\Lambda}$ is well defined and satisfies the following properties:*

- (a) $\tilde{\Lambda}(\emptyset) = 0$.
- (b) $\tilde{\Lambda}(\bigcup_{n \in \mathbb{N}} B_n) = \sum_{n=1}^{\infty} \tilde{\Lambda}(B_n)$ for every countable family of disjoint sets $B_n \in \pi(\mathcal{B}^*, \mathcal{B}_\Lambda)$, $n \in \mathbb{N}$.

PROOF. $\tilde{\Lambda}$ is well defined if and only if the function $h : T \rightarrow \mathbb{R} \cup \{\infty\}$, $h(t) = \#(B \cap (P \times \{t\}))$, is measurable with respect to the σ -algebra \mathcal{B}_Λ in T ; i.e., if $h^{-1}(\{n\}) \in \mathcal{B}_\Lambda$ for all $n \in \mathbb{N}$. To prove this property, we proceed by induction on n . It is clear that $h^{-1}(\{0\}) = T \setminus \pi(B)$ belongs to \mathcal{B}_Λ since $B \in \pi(\mathcal{B}^*, \mathcal{B}_\Lambda)$. Now, suppose $h^{-1}(\{i\}) \in \mathcal{B}_\Lambda$ for $i \in \{0, \dots, n-1\}$ and let us check that $h^{-1}(\{n\}) \in \mathcal{B}_\Lambda$. Let

$$C_n = \{((t, p_1), (t, p_2), \dots, (t, p_{n+1})) \mid t \in T, p_1, \dots, p_{n+1} \in P\},$$

which is a closed in $(P \times T)^{n+1}$. Observe that C_n is the set of $(n+1)$ -uples in $T \times P$ that lie in the same plaque. We remark that C_n is homeomorphic to $P^{n+1} \times \Delta_T$, where Δ_T is the diagonal of the product T^{n+1} , and the projection $\pi_T : \Delta_T \rightarrow T$, $(t, \dots, t) \mapsto t$ is a homeomorphism. The measure Λ becomes a measure on Δ_T via π_T . The intersection $B^{n+1} \cap C_n$, denoted by D_n , is the set of $(n+1)$ -uples in B that lie in the same plaque. Let

$$\Delta_n = \{((t, p_1), (t, p_2), \dots, (t, p_{n+1})) \in C_n \mid \exists i, j \text{ with } i \neq j \text{ and } p_i = p_j\},$$

which is closed in C_n . This set consists of the $(n+1)$ -uples in each plaque such that two components are equal. The set $D_n \setminus \Delta_n$ consists of the $(n+1)$ -uples of different elements in B that lie in the same plaque. Therefore, $\pi_T \circ \pi_\Delta(D_n \setminus \Delta_n)$ consists of the points $t \in T$ such that the corresponding plaque $P \times \{t\}$ contains more than n points of B , where $\pi_\Delta : C_n \rightarrow \Delta_T$ is the natural projection.

Now, let us prove that $\pi_\Delta(D_n \setminus \Delta_n) \in \pi_T^{-1}(\mathcal{B}_\Lambda) = \pi_T^{-1}(\mathcal{B})_\Lambda$. By Lemma 2.3, $\hat{\pi}(B^{n+1} \setminus \Delta_n) \in \langle \mathcal{B}_\Lambda^{n+1} \rangle$, where $\hat{\pi} : (P \times T)^{n+1} \rightarrow T^{n+1}$ is the natural projection. Therefore

$$\pi_\Delta(D_n \setminus \Delta_n) = \Delta_T \cap \hat{\pi}(B^{n+1} \setminus \Delta_n) \in \langle \mathcal{B}_\Lambda^{n+1} \rangle|_{\Delta_T},$$

where $\langle \mathcal{B}_\Lambda^{n+1} \rangle|_{\Delta_T}$ denotes the restriction of the σ -algebra $\langle \mathcal{B}_\Lambda^{n+1} \rangle$ to Δ_T . We only have to prove that $\langle \mathcal{B}_\Lambda^{n+1} \rangle|_{\Delta_T} \subset \pi_T^{-1}(\mathcal{B}_\Lambda)$. For that purpose, we have to check that the generators $\prod_{k=1}^{n+1} F_k$, with $F_k \in \mathcal{B}_\Lambda$, satisfy $(\prod_{k=1}^{n+1} F_k) \cap \Delta_T \in \pi_T^{-1}(\mathcal{B}_\Lambda)$. For each k , take $A_k, B_k \in \mathcal{B}$ with $A_k \subset F_k \subset B_k$ and $\Lambda(B_k \setminus A_k) = 0$. Then

$$\left(\prod_{k=1}^{n+1} A_k \right) \cap \Delta_T \subset \left(\prod_{k=1}^{n+1} F_k \right) \cap \Delta_T \subset \left(\prod_{k=1}^{n+1} B_k \right) \cap \Delta_T,$$

and $\left(\prod_{k=1}^{n+1} A_k \right) \cap \Delta_T$ and $\left(\prod_{k=1}^{n+1} B_k \right) \cap \Delta_T$ belong to $\pi_T^{-1}(\mathcal{B})$ because

$$\left(\prod_{k=1}^{n+1} A_k \right) \cap \Delta_T = \bigcap_{k=1}^{n+1} \pi_T^{-1}(A_k), \quad \left(\prod_{k=1}^{n+1} B_k \right) \cap \Delta_T = \bigcap_{k=1}^{n+1} \pi_T^{-1}(B_k).$$

Moreover

$$\begin{aligned} \Lambda \left(\left(\left(\prod_{k=1}^{n+1} B_k \right) \cap \Delta_T \right) \setminus \left(\left(\prod_{k=1}^{n+1} A_k \right) \cap \Delta_T \right) \right) \\ = \Lambda \left(\bigcap_{k=1}^{n+1} (\pi_T^{-1}(B_k) \setminus \pi_T^{-1}(A_k)) \right) = \Lambda \left(\pi_T^{-1} \left(\bigcap_{k=1}^{n+1} (B_k \setminus A_k) \right) \right) \\ = \Lambda \left(\bigcap_{k=1}^{n+1} (B_k \setminus A_k) \right) = 0 . \end{aligned}$$

This shows that $\pi_\Delta(D_n \setminus \Delta_n) \in \pi_T^{-1}(\mathcal{B}_\Lambda) = \pi_T^{-1}(\mathcal{B})_\Lambda$. By induction, we have

$$h^{-1}(n) = T \setminus ((\pi_T \circ \pi_\Delta(D_n \setminus \Delta_n) \cup h^{-1}(\{0, \dots, n-1\})) \in \mathcal{B}_\Lambda.$$

Property (a) is obvious. To show property (b), observe that $\chi_{\bigcup B_n} = \sum \chi_{B_n}$, and then use the monotonous convergence theorem. \square

DEFINITION 2.6. If $B \in \mathcal{B}^* \setminus \pi(\mathcal{B}^*, \mathcal{B}_\Lambda)$, then define $\tilde{\Lambda}(B) = \infty$.

PROPOSITION 2.7. $(P \times T, \mathcal{B}^*, \tilde{\Lambda})$ is a measure space and $\tilde{\Lambda}$ extends Λ .

PROOF. We only have to prove that $\tilde{\Lambda}(\bigcup_n B_n) = \sum_n \tilde{\Lambda}(B_n)$ for every countable family of disjoint sets $B_n, n \in \mathbb{N}$, in \mathcal{B}^* . By Proposition 2.5, this holds if $B_n \in \pi(\mathcal{B}^*, \mathcal{B}_\Lambda)$ for all $n \in \mathbb{N}$. If $\bigcup_n B_n \in \mathcal{B}^* \setminus \pi(\mathcal{B}^*, \mathcal{B}_\Lambda)$, then the above equality is obvious. So we only have to consider the case where some $B_j \in \mathcal{B}^* \setminus \pi(\mathcal{B}^*, \mathcal{B}_\Lambda)$ and, however, $\bigcup_n B_n \in \pi(\mathcal{B}^*, \mathcal{B}_\Lambda)$. We can suppose $B_j = B_1$, and let $B = \bigcup_n B_n$. Let

$$B^\infty = \{t \in T \mid \#(B \cap (P \times \{t\})) = \infty\},$$

which belongs to \mathcal{B}_Λ by Proposition 2.5. The proof will be finished by checking that $\Lambda(B^\infty) > 0$. We have $B_1^\infty \subset B^\infty$, where

$$B_1^\infty = \{t \in T \mid \#(B_1 \cap (P \times \{t\})) = \infty\}.$$

Suppose $\Lambda(B^\infty) = 0$. Since $B^\infty \in \mathcal{B}_\Lambda$, there is some $A \in \mathcal{B}$ such that $B^\infty \subset A$ and $\Lambda(A) = 0$. The Borel set $\pi^{-1}(A)$ satisfies $B_1 \cap \pi^{-1}(A) \in \pi(\mathcal{B}^*, \mathcal{B}_\Lambda)$ since $\emptyset \subset \pi(B_1 \cap \pi^{-1}(A) \cap U) \subset A$ and $\Lambda(A) = 0$ for each open set $U \subset P \times T$. On the other hand, $B_1 \setminus \pi^{-1}(A)$ is a Borel set meeting every plaque in a finite set, which is σ -compact, and therefore projects to a Borel set by Proposition 2.1. Hence $B_1 \in \pi(\mathcal{B}^*, \mathcal{B}_\Lambda)$ by Proposition 2.2, which is a contradiction. \square

We have constructed an extension of each transverse invariant measure in a product measurable lamination, but its uniqueness was not proved. This uniqueness is false in general. For instance, take the foliated product $\mathbb{R} \times \{*\}$ and let Λ be the null measure on the singleton $\{*\}$; our extension $\tilde{\Lambda}$ is the zero measure in the total space. Now, let μ be the measure defined by

- (i) $\mu(B) = 0$ for all countable set B ; and
- (ii) $\mu(B) = \infty$ for all uncountable Borel set B .

This measure μ extends Λ too and is quite different from $\tilde{\Lambda}$. In order to solve this problem, we require some conditions to the extension. These conditions have the spirit of coherency with the concept of transverse invariant measures. We will prove that our extension is the unique coherent extension.

DEFINITION 2.8. Let μ be an extension of a transverse invariant measure Λ on $P \times T$. The measure μ is called a *coherent extension* of Λ if satisfies the following conditions:

- (a) If $B \in \mathcal{B}^*$, $B \not\subset \pi^{-1}(S)$ for any $S \in \mathcal{B}$ with $\Lambda(S) = 0$, and $\#B \cap P \times \{t\} = \infty$ for each plaque $P \times \{t\}$ which meets B , then $\mu(B) = \infty$.
- (b) If $\Lambda(S) = 0$ for some $S \in \mathcal{B}$, then $\mu(\pi^{-1}(S)) = 0$.
- (c) If $B \in \mathcal{B}^*$ and $\Lambda(S) = \infty$ for all $S \in \mathcal{B}$ with $B \subset \pi^{-1}(S)$, then $\mu(B) = \infty$.

REMARK 4. Condition (a) determines μ on Borel sets with infinite points in plaques which are not contained in the saturation of a Λ -null set. Condition (b) means certain coherency between the support of Λ and the support of the extension μ . Condition (c) determines μ on any Borel set so that any Borel set containing its projection has infinity Λ -measure.

PROPOSITION 2.9. $\tilde{\Lambda}$ is the unique coherent extension.

PROOF. We prove that every coherent extension has the same values as $\tilde{\Lambda}$ on \mathcal{B}^* . First, we consider the case $B \in \pi(\mathcal{B}^*, \mathcal{B}_\Lambda)$. Let

$$B^\infty = \{t \in T \mid \#(B \cap (P \times \{t\})) = \infty\}.$$

This set belongs to \mathcal{B}_Λ by Proposition 2.5. Therefore there exist Borel sets A, C such that $A \subset B^\infty \subset C$ and $\Lambda(C \setminus A) = 0$. Let $\tilde{B}^\infty = B \cap \pi^{-1}(C)$. The Borel set $B \setminus \tilde{B}^\infty$ is a generalized transversal and hence $\mu(B \setminus \tilde{B}^\infty) = \tilde{\Lambda}(B \setminus \tilde{B}^\infty)$. On the other hand, if $\Lambda(\pi(\tilde{B}^\infty)) = 0$, then $\Lambda(C) = 0$ and $\mu(\tilde{B}^\infty) \leq \mu(\pi^{-1}(C)) = 0$ by (b). If $\Lambda(\pi(\tilde{B}^\infty)) > 0$, let $\hat{B}^\infty = B \cap \pi^{-1}(A)$ and $B'^\infty = B \cap \pi^{-1}(C \setminus A)$. Then $\mu(\hat{B}^\infty) = \infty$ by (a), and $\mu(B'^\infty) = 0$ by (b). Therefore μ equals $\tilde{\Lambda}$ on $\pi(\mathcal{B}^*, \mathcal{B}_\Lambda)$.

The case $B \in \mathcal{B}^* \setminus \pi(\mathcal{B}^*, \mathcal{B}_\Lambda)$ is similar. The set B^∞ is not a Borel set in this case, but observe that $(B \cap \pi^{-1}(B^\infty)) \not\subset \pi^{-1}(S)$ with $\Lambda(S) < \infty$ or we obtain $\pi(B \cap \pi^{-1}(S) \cap U) \in \mathcal{B}_\Lambda$ for all open set $U \subset P \times T$ by Remark 1, since $B \cap \pi^{-1}(S) \cap U$ is a Borel set in $P \times S$ and Λ is finite in S . Hence $B \in \pi(\mathcal{B}^*, \mathcal{B}_\Lambda)$ by Propositions 2.2 and 2.1. Therefore $\mu(B) = \infty$ by (c). This proves that μ and $\tilde{\Lambda}$ agree on \mathcal{B}^* , as desired. \square

2. The general case

In this section, we prove the following theorem.

THEOREM 2.10. Let X be a measurable Polish lamination with a transverse invariant measure Λ . There exists a measure $\tilde{\Lambda}$ on X which is the unique coherent extension of Λ .

Let $\{U_i, \varphi_i\}_{i \in \mathbb{N}}$ be a foliated measurable atlas with $\varphi_i(U_i) = \mathbb{R}^n \times T_i$, where T_i is a standard Borel space. It is clear that $\varphi_i^{-1}(\{*\} \times T_i)$ is a generalized transversal and, via φ_i , we obtain a Borel measure Λ_i on T_i . Proposition 2.7 provides a measure $\tilde{\Lambda}_i$ on $U_i \approx \mathbb{R}^n \times T_i$ that extends Λ_i .

Moreover Proposition 2.1 gives $\tilde{\Lambda}_i(T) = \Lambda(T)$ for all generalized transversal $T \subset U_i$. Let $\pi_i : U_i \rightarrow \varphi_i^{-1}(\{*\} \times T_i)$ denote the natural projections.

LEMMA 2.11. *Let B be a Borel subset of $U_i \cap U_j$, $i, j \in \mathbb{N}$. Then*

$$B \in \pi_i(\mathcal{B}^*, \mathcal{B}_{\Lambda_i}) \iff B \in \pi_j(\mathcal{B}^*, \mathcal{B}_{\Lambda_j}).$$

PROOF. By Lemma 1.3, there exists a countable family of measurable holonomy transformations from $\varphi_i^{-1}(\{*\} \times T_i)$ to $\varphi_j^{-1}(\{*\} \times T_j)$ whose domains and ranges cover $\pi_i(U_i \cap U_j)$ and $\pi_j(U_i \cap U_j)$, respectively. Therefore, if A is a Borel set contained in $U_i \cap U_j$ and $\pi_i(A)$ is a Borel set, then $\pi_j(A)$ is a Borel set and

$$\Lambda(\pi_i(A)) = 0 \iff \Lambda(\pi_j(A)) = 0.$$

□

LEMMA 2.12. *$\tilde{\Lambda}_i(B) = \tilde{\Lambda}_j(B)$ for all Borel set $B \subset U_i \cap U_j$, $i, j \in \mathbb{N}$.*

PROOF. We remark that $\tilde{\Lambda}_i$ and $\tilde{\Lambda}_j$ have the same values in generalized transversals of $U_i \cap U_j$. By Lemma 2.11, we only consider Borel sets in $\pi_i(\mathcal{B}^*, \mathcal{B}_{\Lambda})$. Suppose that $\pi_i(B)$ is a Borel set; otherwise, $\pi_i(B)$ is Λ -measurable and we can choose a Borel set $A \subset \pi_i(B)$ with $\Lambda(\pi_i(B) \setminus A) = 0$. We take the Borel set $\tilde{B} = B \cap \pi_i^{-1}(A)$. This Borel set projects onto the Borel set A and $\tilde{\Lambda}_i(B \setminus \tilde{B}) = \tilde{\Lambda}_j(B \setminus \tilde{B}) = 0$ by Definition 2.4 and Lemma 2.11, hence $\tilde{\Lambda}_i(B) = \tilde{\Lambda}_i(\tilde{B})$ and $\tilde{\Lambda}_j(B) = \tilde{\Lambda}_j(\tilde{B})$. Let

$$B^k = \{t \in T_i \mid \#(\varphi_i(B) \cap (\mathbb{R}^n \{t\})) = k\}, \quad k \in \mathbb{N} \cup \{\infty\}.$$

These are Λ -measurable sets by Proposition 2.5, and we assume that they are Borel sets by the same reason as above. Let \tilde{B}^k denote $B \cap \pi_i^{-1}(\varphi_i^{-1}(\{*\} \times B_k))$, which is a Borel set. It is obvious that $\bigcup_{k=1}^{\infty} \tilde{B}^k$ is a generalized transversal, hence $\tilde{\Lambda}_i(\bigcup_{k=1}^{\infty} \tilde{B}^k) = \tilde{\Lambda}_j(\bigcup_{k=1}^{\infty} \tilde{B}^k)$. Now consider

$$\tilde{B}_l^{\infty} = \{x \in \tilde{B}^{\infty} \mid \#(B \cap P_x^j) = l\}, \quad l \in \mathbb{N} \cup \{\infty\},$$

where P_x^j denotes the plaque of U_j that contains x . The proof is finished in the case $\Lambda(\pi_i(\tilde{B}^{\infty})) = 0$ (we can restrict to the case of a generalized transversal). If $\Lambda(\pi_i(\tilde{B}^{\infty})) > 0$, then $\Lambda(\pi_j(\tilde{B}^{\infty})) > 0$. Therefore we obviously obtain $\tilde{\Lambda}_i(B) = \infty = \tilde{\Lambda}_j(B)$. □

DEFINITION 2.13. Let B be a measurable set in X , and

$$B_1 = B \cap U_1, \quad B_k = (B \cap U_k) \setminus (B_1 \cup \dots \cup B_{k-1}),$$

for $k \geq 2$. Define

$$\tilde{\Lambda}(B) = \sum_{i=1}^{\infty} \tilde{\Lambda}_i(B_i).$$

By Lemma 2.12, it is easy to prove that Definition 2.13 does not depend neither on the ordering of the charts nor on the choice of the countable foliated measurable atlas. It is also easy to prove that $\tilde{\Lambda}$ extends Λ since both of them have the same values on generalized transversals contained in each chart and, hence, in every generalized transversal. Theorem 2.10 is now established.

DEFINITION 2.14. Let μ be an extension of a transverse invariant measure Λ on a measurable lamination (X, \mathcal{F}) . The measure μ is called a *coherent extension* of Λ if it is a coherent extension on each foliated measurable chart with the induced transverse invariant measure.

COROLLARY 2.15. *The extension $\tilde{\Lambda}$ is the unique coherent extension of Λ .*

Theorem 2.10 gives a new interpretation of transverse invariant measures. It can be also used to introduce the following version of the concept of transversal for measurable laminations with transverse invariant measures.

DEFINITION 2.16. Let X be a measurable lamination with a transverse invariant measure Λ . A Borel subset of X with finite $\tilde{\Lambda}$ -measure is called a *Λ -generalized transversal*.

CHAPTER 3

LS category on measurable laminations

1. Measurable tangential category and Λ -category

A measurable Polish lamination (X, \mathcal{F}) induces a foliated measurable structure \mathcal{F}_U in each measurable open set U (by Theorem 1.2). The space $U \times \mathbb{R}$ admits an obvious foliated structure $\mathcal{F}_{U \times \mathbb{R}}$ whose leaves are products of leaves of \mathcal{F}_U and \mathbb{R} . An MT-map $H : \mathcal{F}_{U \times \mathbb{R}} \rightarrow \mathcal{G}$ (where (Y, \mathcal{G}) is a measurable Polish lamination) is called a *(measurable) tangential homotopy*, and it is said that the maps $H(-, 0)$ and $H(-, 1)$ are *(measurably) homotopic*. We use the term *(measurable) tangential deformation* when $\mathcal{G} = \mathcal{F}$ and $H(-, 0)$ is the inclusion map of U . A tangential deformation such that $H(-, 1)$ is constant on the leaves of \mathcal{F}_U is called a *(measurable) tangential contraction* or an *\mathcal{F} -contraction*, and U is called a *tangentially categorical* or *\mathcal{F} -categorical* open set. All of these definitions have obvious versions for foliated spaces by changing (ambient space) measurability for continuity.

The *tangential category* is the lowest number of tangentially categorical open sets that cover the measurable Polish lamination. On one leaf foliations, this definition agree with the classical category. The tangential category of \mathcal{F} is denoted by $\text{Cat}(\mathcal{F})$. It is clear that it is a tangential homotopy invariant.

LEMMA 3.1 (Kallman [22]). *Let $P \times T$ be a product of a Polish space P with a standard space T , and let $\pi : P \times T \rightarrow T$ the second factor projection. Let $B \subset P \times T$ be a measurable subset such that $B \cap (P \times \{t\})$ is σ -compact for all $t \in T$. Then $\pi(B)$ is measurable.*

PROPOSITION 3.2. *For any measurable open set U and any tangential deformation H , the set $H(U \times \{1\})$ is measurable.*

PROOF. By the Kunugui-Novikov's theorem (Theorem 1.2), there exist countable families, $\{P_n \times T_n\}$ and $\{P'_n \times T'_n\}$ ($n \in \mathbb{N}$), and MT-embeddings $f_n : P_n \times T_n \rightarrow \mathcal{F}$ and $g_n : P'_n \times T'_n \rightarrow \mathcal{F}$ such that $U = \bigcup_n f_n(P_n \times T_n)$ and $H(-, 1) \circ f_n(P_n \times T_n) \subset g_n(P'_n \times T'_n)$. Hence we only have to prove that each $g_n^{-1} \circ H(-, 1) \circ f_n(P_n \times T_n)$ is measurable. Consider a topology on T_n so that it is isomorphic to $[0, 1]$, \mathbb{N} or a finite set [23], and let $P_n \times T_n$ be endowed with the product topology τ , becoming a Polish space. Let $\pi : (P_n \times T_n, \tau) \times P'_n \times T'_n \rightarrow P'_n \times T'_n$ be the second factor projection. For each point $(x, t) \in P'_n \times T'_n$, the set $f_n^{-1} \circ H(-, 1)^{-1} \circ g_n(x, t)$ is σ -compact in $(P_n \times T_n, \tau)$. By the continuity of $H(-, 1) \circ f_n$ (with the leaf topology), this preimage is closed on each plaque $P_n \times \{t\}$ ($t \in T_n$). On the other hand, this preimage only cuts a countable number of leaves (otherwise H would not be a tangential deformation), and therefore it only cuts a countable number of

plaques. Hence this preimage is also σ -compact with the topology τ . Let

$$B_n = \{ ((x, t), (x', t')) \mid g_n^{-1} \circ H(-, 1) \circ f_n(x, t) = (x', t') \},$$

which is measurable in $(P_n \times T_n, \tau) \times P'_n \times T'_n$. Then, by Lemma 3.1, $\pi(B_n)$ is measurable. But $\pi(B_n) = g_n^{-1} \circ H(-, 1) \circ f_n(P_n \times T_n)$. \square

Let Λ be a transverse invariant measure for \mathcal{F} . Define

$$\tau_\Lambda(U) = \inf \{ \tilde{\Lambda}(H(U \times 1)) \mid H \text{ tangential deformation on } U \},$$

where $\tilde{\Lambda}$ denotes the coherent extension of Λ . Then the Λ -category of (\mathcal{F}, Λ) is defined as

$$\text{Cat}(\mathcal{F}, \Lambda) = \inf_{\mathcal{U}} \sum_{U \in \mathcal{U}} \tau_\Lambda(U),$$

where \mathcal{U} runs in the countable coverings of X by measurable open sets. The countability condition of the coverings is needed, otherwise Λ -category would be directly zero when Λ has no atoms. When the homotopies used in this definition are required to be C^r on leaves, we use the notation $\text{Cat}^r(\mathcal{F}, \Lambda)$ and the term C^r Λ -category.

2. Homotopy invariance and other properties

An MT-homotopy equivalence h from \mathcal{F} to \mathcal{G} induces a bijection between the sets of transverse invariant measures on \mathcal{G} and \mathcal{F} . Its definition is given as follows. Let T be a complete transversal of \mathcal{F} . Obviously $h|_T$ has countable fibers. By Proposition 1.1, $h(T)$ is a transversal of \mathcal{G} . In fact, there exists a countable measurable partition, $T = \bigcup_i T_i$, so that each $h|_{T_i}$ is injective. Define $h^* \Lambda(T) = \sum_i \Lambda(h(T_i))$. In the following, we always suppose that an MT-homotopy equivalence between measurable laminations with transverse invariant measures must be compatible with the measures in this sense.

LEMMA 3.3. *Let $(X, \mathcal{F}, \Lambda)$ and (Y, \mathcal{G}, Δ) be measurable laminations with transverse invariant measures, let $h : (X, \Lambda) \rightarrow (Y, \Delta)$ be a measurable homotopy equivalence. Then, for all measurable set $K \subset X$ with σ -compact intersections with the leaves, $h(K)$ is measurable and $\tilde{\Delta}(h(K)) \leq \tilde{\Lambda}(K)$.*

PROOF. The fact that $h(K)$ is measurable is a consequence of Proposition 3.2. Notice that finite intersections of σ -compact sets are σ -compact. Let $\mathcal{U} = \{(U_n, \varphi_n)\}$ and $\mathcal{V} = \{(V_n, \psi_n)\}$ ($n \in \mathbb{N}$) be foliated measurable atlases for \mathcal{F} and \mathcal{G} , respectively. Observe that there exists a foliated measurable atlas $\mathcal{W} = \{(W_n, \phi_n)\}$ on X satisfying the following conditions:

- (a) For each $n \in \mathbb{N}$, there exists some $k(n) \in \mathbb{N}$ such that $h(W_n) \subset V_{k(n)}$.
- (b) The map h induces an injective map from the family of plaques of W_n to the family of plaques of $V_{k(n)}$; i.e., each plaque of $V_{k(n)}$ contains at most the image by h of one plaque of W_n .

This atlas can be easily obtained by using Theorem 1.2 and Proposition 1.1.

The maps $\psi_{k(m)} \circ h \circ \phi_m^{-1} : P_m \times T_m \rightarrow P'_{k(m)} \times T'_{k(m)}$ are injective in the plaques in the sense of (b). Clearly, $\Lambda(D) = \Delta(h(D))$ for all Borel sets

$D \subset T_m$. Let P^t be the plaque of $P'_{k(m)} \times T'_{k(m)}$ that contains $\psi_{k(m)} \circ h \circ \phi_m^{-1}(P_m \times \{t\})$, for $t \in T_m$. We have

$$\#(h(K) \cap P^t) \leq \#(K \cap (P \times \{t\}))$$

for every $t \in T_m$. Hence

$$\begin{aligned} \tilde{\Delta}(\psi_{k(m)} \circ h \circ \phi_m^{-1}(K)) &= \int_{T'_{k(m)}} \#(\psi_{k(m)} \circ h \circ \phi_m^{-1}(K) \cap (P' \times \{t'\})) d\Delta(t') \\ &= \int_{\psi_{k(m)} \circ h \circ \phi_m^{-1}(T_m)} \#(\psi_{k(m)} \circ h \circ \phi_m^{-1}(K) \cap (P \times \{t'\})) d\Delta(t') \\ &= \int_{T_m} \#(\psi_{k(m)} \circ h \circ \phi_m^{-1}(K) \cap P^t) d\Lambda(t) \\ &\leq \int_{T_m} \#(K \cap (P \times \{t\})) d\Lambda(t) \\ &= \tilde{\Lambda}(K). \end{aligned}$$

Then the Lemma holds on each chart of \mathcal{W} . Consider the family $\{B_k\}$ inductively defined by

$$B_1 = (K \cap W_1), \quad B_k = (K \cap W_k) \setminus (B_1 \cup \dots \cup B_{k-1}) \quad (k > 1).$$

It is a Borel partition of K and its elements have a σ -compact intersection with each leaf. Finally,

$$\tilde{\Delta}(h(K)) = \tilde{\Delta}\left(\bigcup_i h(B_i)\right) \leq \sum_i \tilde{\Delta}(h(B_i)) \leq \sum_i \tilde{\Lambda}(B_i) = \tilde{\Lambda}(K). \quad \square$$

PROPOSITION 3.4 (The Λ -category is an MT-homotopy invariant). *Let $(X, \mathcal{F}, \Lambda)$ and (Y, \mathcal{G}, Δ) be measurable homotopy equivalent measurable laminations with transverse invariant measure. Then $\text{Cat}(\mathcal{F}, \Lambda) = \text{Cat}(\mathcal{G}, \Delta)$.*

PROOF. Let $h : X \rightarrow Y$ be a measurable homotopy equivalence, and let g be a homotopy inverse of h . Let $\{U_n\}$ ($n \in \mathbb{N}$) be a covering of Y by measurable open sets. Then $\{h^{-1}(U_n)\}$ is a covering of X by measurable open sets. We will prove that $\tau_\Lambda(h^{-1}(U_n)) \leq \tau_\Delta(U_n)$ for all $n \in \mathbb{N}$. Let H^n be a measurable tangential deformations on each U_n , and F an MT-homotopy connecting the identity map and $g \circ h$. Let $G : h^{-1}(U) \times \mathbb{R} \rightarrow X$ be the MT-homotopy

$$h^{-1}(U) \times \mathbb{R} \xrightarrow{h \times \text{id}} U \times [0, 1] \xrightarrow{H} Y \xrightarrow{g} X.$$

Then $K : h^{-1}(U) \times \mathbb{R} \rightarrow X$, defined by

$$K(x, t) = \begin{cases} F(x, 2t) & \text{if } t \leq 1/2 \\ G(x, 2t - 1) & \text{if } t \geq 1/2, \end{cases}$$

where F is the MT-homotopy from the identity map on X to $g \circ h$, is a tangential deformation. Lemma 3.3 yields

$$\tilde{\Lambda}(K(h^{-1}(U) \times \{1\})) = \tilde{\Lambda}(g(H(U \times \{1\}))) \leq \tilde{\Delta}(H(U \times \{1\})).$$

Hence $\tau_\Lambda(h^{-1}(U_n)) \leq \tau_\Lambda(U_n)$ for all $n \in \mathbb{N}$. Therefore $\text{Cat}(\mathcal{F}, \Lambda) \leq \text{Cat}(\mathcal{G}, \Delta)$. The inverse inequality is analogous. \square

The above proposition has an obvious C^r version. The tangential category and the Λ -category are connected by the following result.

PROPOSITION 3.5. *Let $(X, \mathcal{F}, \Lambda)$ be a measurable lamination with a transverse invariant measure, and let U be a measurable open set in X . If $\tau_\Lambda(U) < \infty$, then there exists a tangentially categorical open set $U' \subset U$ such that $\tilde{\Lambda}(U \setminus U') = 0$.*

PROOF. There exists a tangential deformation of U such that $\tilde{\Lambda}(H(U \times \{1\})) < \infty$. This means, by the conditions of a coherent extension, that $H(U \times \{1\}) = B \cup T$, where B is a Borel set such that $\tilde{\Lambda}(B) = 0$ and T is a transversal of \mathcal{F} . The set $U' = H(-, 1)^{-1}(T)$ satisfies the required conditions. \square

DEFINITION 3.6. Let $(X, \mathcal{F}, \Lambda)$ be a measurable lamination with a transverse invariant measure. A *null-transverse* set is a measurable set B such that $\tilde{\Lambda}(B) = 0$.

The following propositions are elementary.

PROPOSITION 3.7. *Let $(X, \mathcal{F}, \Lambda)$ be a measurable lamination with a transverse invariant measure, and let B be a null-transverse set. Then $\text{Cat}(\mathcal{F}, \Lambda)$ can be computed by using only coverings of $X \setminus B$ or $X \setminus \text{sat}(B)$. If B is saturated, then*

$$\text{Cat}(\mathcal{F}, \Lambda) = \text{Cat}(\mathcal{F}|_{X \setminus B}, \Lambda|_{X \setminus B}) .$$

PROPOSITION 3.8. *Let T be a measurable transversal which meets each leaf at most in one point. Then $\Lambda(T) \leq \text{Cat}(\mathcal{F}, \Lambda)$.*

PROPOSITION 3.9. *Let $(X, \mathcal{F}, \Lambda)$ be a measurable lamination with a transverse invariant measure Λ . Suppose that (X, \mathcal{F}) is defined by a measurable suspension $\tilde{M} \times_h S$. Then $\text{Cat}(\mathcal{F}, \Lambda) \leq \text{Cat}(M) \cdot \Lambda(S)$.*

PROPOSITION 3.10. *For a manifold M and a standard Borel space T , let $M \times T$ be foliated as a product. Then, for every measure Λ on T , considered as an invariant measure Λ on $M \times T$, $\text{Cat}(M \times T, \Lambda) = \text{Cat}(M) \cdot \Lambda(T)$.*

PROPOSITION 3.11. *Let $\{U_n\}$ ($n \in \mathbb{N}$) be a covering by saturated measurable open sets of $(X, \mathcal{F}, \Lambda)$. Then*

$$\text{Cat}(\mathcal{F}, \Lambda) \leq \sum_{n \in \mathbb{N}} \text{Cat}(\mathcal{F}|_{U_n}, \Lambda) .$$

Here, the equality holds if $\{U_n\}$ is a partition.

DEFINITION 3.12. Let $U \subset (X, \mathcal{F})$ be a measurable open set. Define the relative tangential category of U by $\text{Cat}(U, \mathcal{F}) = \min_{\mathcal{U}} \#\mathcal{U}$, where \mathcal{U} runs in the family of measurable open coverings of U by tangentially categorical measurable open sets. When \mathcal{F} has a transverse invariant measure Λ , the relative Λ -category of U is defined by

$$\text{Cat}(U, \mathcal{F}, \Lambda) = \inf_{\mathcal{U}} \sum_{V \in \mathcal{U}} \tau_\Lambda(V) ,$$

where \mathcal{U} runs in the family of countable measurable open coverings of U .

REMARK 5. Observe that $\tau_\Lambda(V)$ is defined by using measurable tangential homotopies deforming V in the ambient space. Clearly, $\text{Cat}(U, \mathcal{F}, \Lambda) \leq \text{Cat}(\mathcal{F}_U, \Lambda)$. The same is true for the relative tangential category.

PROPOSITION 3.13 (Subadditivity of relative Λ -category). *Let $\{U_i\}$ ($i \in \mathbb{N}$) be a countable family of measurable open subsets of X . Then*

$$\text{Cat}\left(\bigcup_{i \in \mathbb{N}} U_i, \mathcal{F}, \Lambda\right) \leq \sum_{i \in \mathbb{N}} \text{Cat}(U_i, \mathcal{F}, \Lambda).$$

The same is true for the relative tangential category.

3. Measurable category of laminations with compact leaves

In this section, we compute the measurable Λ -category on foliated spaces with all leaves compact. With these conditions, there exists a countable filtration

$$\cdots \subset E_\alpha \subset \cdots \subset E_2 \subset E_1 \subset E_0 = X,$$

such that each E_α is a closed saturated set, and $E_\alpha \setminus E_{\alpha+1}$ is dense in E_α and consists of leaves with trivial holonomy on the foliated space E_α , remark that the order induced in the filtration may not agree with the usual order of \mathbb{N} . This family will be called the *Epstein filtration* of X [14, 15, 12].

Obviously, each $E_\alpha \setminus E_{\alpha+1}$ is a saturated measurable open set in the measurable sense without holonomy. Hence, by Proposition 7.11,

$$\text{Cat}_{\text{meas}}(\mathcal{F}, \Lambda) = \sum_{\alpha} \text{Cat}_{\text{meas}}(\mathcal{F}|_{E_\alpha \setminus E_{\alpha+1}}, \Lambda),$$

where the subindex *meas* indicates that the tangential category is computed in a measurable way.

THEOREM 3.14 (See e.g. [35]). *Let R be an equivalence relation on a Polish space X such that every equivalence class is a closed set in X . If the saturation of open sets of X are Borel, then there exists a Borel set meeting all equivalence classes in one point. If the saturation of open sets are open, then there exists a Polish subspace meeting all equivalence classes in one point.*

COROLLARY 3.15. *Let (X, \mathcal{F}) be a lamination with all leaves compact and let T be a complete transversal of \mathcal{F} . Then there exists a Polish subspace contained in T meeting every leaf in one point.*

Let Γ be the holonomy pseudogroup of \mathcal{F} on T . Let R be a Polish set satisfying the hypothesis of Theorem 3.14 for the leaf relation induced on T . There exists a bijective map $\pi : R \rightarrow T/\Gamma \approx X/\mathcal{F}$ induced by the projection to the leaf space. This map is measurable with respect to the Borel σ -algebra of the leaf space. By using the Epstein filtration of (X, \mathcal{F}) , it is easy to see that π is a Borel isomorphism, even when the leaf space is not Hausdorff. If Λ is a transverse invariant measure, then it induces a measure on R since it is a Borel transversal. Hence Λ induces a measure $\Lambda_{\mathcal{F}}$ on the leaf space via the Borel isomorphism π . The measure $\Lambda_{\mathcal{F}}$ is independent of the choice of the Polish (Borel) sets R and T , since all of them are equivalent by a measurable holonomy map.

REMARK 6. By using the Epstein filtration, it easily follows that the category map, $\text{Cat} : (X/\mathcal{F}, \mathcal{M}_\pi) \rightarrow \mathbb{N} \cup \{\infty\}$, assigning to each leaf its category, is measurable.

The computation of the measurable Λ -category of a foliated space with all leaves compact can be restricted to the case with trivial holonomy as we have seen. X has at most a countable number of connected components. If all leaves have trivial holonomy, then the leaves on each connected component are homeomorphic. So, in general, $\mathcal{O}(\mathcal{F})$ is MT-isomorphic to the disjoint union of product spaces $L_n \times R_n$, where L_n are the generic leaves in each connected component and R_n are Borel sets meeting leaves in one point in these components. By Propositions 7.10 and 7.11, we obtain the following corollary.

COROLLARY 3.16. *Let $(X, \mathcal{F}, \Lambda)$ be a foliated space with compact leaves endowed with a transverse invariant measure. Then*

$$\text{Cat}(\mathcal{O}(\mathcal{F}), \Lambda) = \int_{X/\mathcal{F}} \text{Cat}(L) d\Lambda_{\mathcal{F}}(L) \leq \text{Cat}(\mathcal{F}, \Lambda) .$$

REMARK 7. This example contains the case of a measurable lamination with a transverse invariant measure supported on a countable number of compact leaves by using Proposition 7.7. Here, the Λ -category agrees with the Λ -mean of the LS category of the compact leaves.

By using the same decomposition, it is easy to check that the measurable tangential category is exactly the maximum of the LS-category of the leaves, $\text{Cat}(\mathcal{O}(\mathcal{F})) = \max\{\text{Cat}(L) \mid L \in \mathcal{F}\}$.

CHAPTER 4

Dimensional upper bound for the measurable Λ -category

It is known that $\text{Cat}(M) \leq \dim M + 1$ for any manifold M [21]. This result was adapted to tangential category in C^2 -foliations on closed manifolds by E. Vogt and W. Singhoff [33], $\text{Cat}(\mathcal{F}) \leq \dim \mathcal{F} + 1$. We show an adaptation to our measurable setting.

1. Measurable triangulations and other considerations

Suppose that there exists a complete Riemannian metric g on the leaves of a C^1 -measurable lamination which varies in a measurable way on the ambient space.

LEMMA 4.1. *The function $i : X \rightarrow \mathbb{R}^+ \cup \{\infty\}$, defined as the injectivity radius of the exponential map at each point, is measurable.*

PROOF. Let $\mathcal{U} = \{U_l\}_{l \in \mathbb{N}}$ be a regular measurable atlas, where $U_l \cong B_l \times T_l$. Clearly, i is measurable on the leaves since the injectivity radius map is lower semicontinuous for each Riemannian manifold. Consider the Borel σ -algebra associated to the compact-open topology in $C(B_l, \mathbb{R}^{m^2})$. Then the Riemannian metric g on the chart U_l can be considered as a measurable map $g : T \rightarrow C(B_l, \mathbb{R}^{m^2})$, where $g(t)(x)$ is the matrix of coefficients of g at (x, t) with respect to the canonical frame of $T\mathbb{R}^m$. In fact, we can work with the closure $\overline{B}_l \times T$. Let $\{\{U(p_i^n)\}_{i \in \mathbb{N}}\}_{n \in \mathbb{N}}$ be a sequence of open coverings of $C(\overline{B}_l, \mathbb{R}^{m^2})$, where $p_i^n \in C(\overline{B}_l, \mathbb{R}^{m^2})$, and $U(p_i^n)$ consists of the functions $f \in C(\overline{B}_l, \mathbb{R}^{m^2})$ such that $\|f - p_i^n\| < 2^{-n}$, using the norm of the maximum absolute value of the coefficients in \mathbb{R}^{m^2} . Therefore $T_i^n = \{g^{-1}(U(p_i^n))\}_{i \in \mathbb{N}}$ is a covering of T by measurable sets. By definition, for $t, t' \in T_i^n$, $\|g(x, t) - g(x, t')\| < 2^{-n}$ for all $x \in \overline{B}_l$.

Let U_{l_1}, \dots, U_{l_N} be a finite sequence of measurable charts (it is possible that $U_{l_i} = U_{l_j}$ for some $i \neq j$) and let $U = \bigcup_{j=1}^N U_{l_j}$. Then U can be decomposed into a countable family of product foliations \mathcal{F}_n such that $\mathcal{F}_n \cap U_{l_j}$ is saturated in U_{l_j} for each j . Of course, we can do the previous argument on each product foliation of the given decomposition.

Since the family of finite collections of measurable charts is countable, we have proved that the lamination \mathcal{F} is a countable union of products $\{K_i^n \times T_i^n\}_{i \in \mathbb{N}}$, where each K_i^n is compact, $\|g(x, t) - g(x, t')\| < 2^{-n}$ for all $x \in K_i^n$ and $t, t' \in T_i^n$, and, for each $x \in \mathcal{F}$, there exists an expansive sequence $K_{i_x^n}^n \subset K_{i_x^{n+1}}^{n+1} \subset \dots$ in L_x meeting x such that $\bigcup_{n \in \mathbb{N}} K_{i_x^n}^n = L_x$, where L_x is the leaf through x . This final property is a consequence of the

fact that these K_i^n are finite unions of closures of plaques in chains of charts associated to each finite sequence of charts in \mathcal{U} .

Finally, the injectivity radius map is measurable by the lower semicontinuity relative to continuous variation of the metric. For n large enough, the Riemannian metric on the plaques of the products $K_i^n \times T_i^n$ is as close to each other as desired. Let $x \in X$ be a point where the injectivity radius is greater than $r \in \mathbb{R}$. By the lower semicontinuity, choose n such that $x \equiv (x_0, t_0) \in \text{int}(K_i^n \times T_i^n)$ and the injectivity radius of each point (y, t) , $(y, t) \in B_x \times T_i^n$, is also greater than r ; where $B_x \subset K_i^n$ is an open neighborhood of x_0 . Since the sequences of products $\{K_i^n \times T_i^n\}_{n,i \in \mathbb{N}}$ form a countable collection and the leaves are second countable, it follows that $i^{-1}(r, \infty]$ is measurable for all $r \in \mathbb{R}$. \square

DEFINITION 4.2 (Measurable triangulation [4]). Let Δ^n denote the canonical n -simplex. A *measurable triangulation* is a family of triangulations on the leaves, $\{\mathcal{T}_L\}_{L \in \mathcal{F}}$, which is *measurable* in the following sense:

- the sets of barycenters of n -simplices, \mathcal{B}^n , are transversals of \mathcal{F} ; and
- the maps $\sigma^n : \Delta^n \times \mathcal{B}^n \rightarrow X$ are measurable, where $\sigma^n(p, -) : \Delta^n \rightarrow L_p$ is the simplex of \mathcal{T}_{L_p} with barycenter p .

A measurable triangulation is of class m if the functions σ_p^n are C^m .

2. Dimensional upper bound

We work in this section with C^1 -measurable triangulations.

Let \mathcal{T} be a measurable triangulation. Let \mathcal{T}^n denote the set of n -faces of \mathcal{T} (the n -simplices of \mathcal{T} without their boundaries).

PROPOSITION 4.3. *Let T be an isolated transversal. There exists a measurable open neighborhood $U(T)$ of T such that the closures of the connected components are disjoint and contains only one point of T . In fact, $U(T)$ can be contracted to T in a measurable way.*

PROOF. Since T is isolated and Borel, the function $D : T \rightarrow \mathbb{R} \cup \{\infty\}$ defined by

$$h(p) = \inf\{d_g(p, p') \mid p' \in T \cap L_p \setminus \{p\}\}$$

is measurable and $h(p) > 0$ for all $p \in T$, where L_p denotes the leaf meeting p and d_g is the distance map on leaves induced by the metric g . The set $h^{-1}(\infty)$ is measurable. Redefine h in this set by $h(p) = 1$, and hence $0 < h < \infty$. Now, let

$$U(T) = \bigcup_{p \in T} B^g(p, \min\{h(p), i(p)\}/2),$$

where $B^g(p, \varepsilon)$ is the d_g -ball in L_p with center on p and radius ε . This set is measurable since h and i are measurable. Obviously, the connected components are the balls $B^g(p, \min\{h(p), i(p)\}/2)$ and satisfy the required conditions. A measurable contraction to T is given by the radial contraction on the tangent space via the exponential map. \square

DEFINITION 4.4. Let $H : U \times \mathbb{R} \rightarrow X$ and $G : V \times \mathbb{R} \rightarrow X$ be tangential deformations such that $H(U \times \{1\}) \subset V$. Let $H * G : U \times \mathbb{R} \rightarrow X$ be the tangential deformation defined by

$$H * G(x, t) = \begin{cases} H(x, 2t) & \text{if } t \leq \frac{1}{2} \\ G(H(x, 1), 2t - 1) & \text{if } \frac{1}{2} \leq t. \end{cases}$$

LEMMA 4.5. Let T be a standard Borel space, let $\mathbb{R}^m \times T$ be endowed with the usual MT-structure and let $\pi : \mathbb{R}^m \times T \rightarrow T$ be the canonical projection. Let S be a transversal that meets each plaque of $\mathbb{R}^n \times T$ at most in one point. Then there exists a measurable homotopy $H : S \times \mathbb{R} \rightarrow \mathbb{R}^m \times T$ such that $H(s, 0) = s$ and $H(s, 1) = (0, \pi(s))$.

PROOF. Consider the measurable homotopy

$$G : (\mathbb{R}^m \times T) \times \mathbb{R} \rightarrow \mathbb{R}^m \times T, \quad ((v, t), s) \mapsto ((1 - s)v, t).$$

Then $H = G|_{S \times \mathbb{R}}$ satisfies the conditions of the statement. \square

COROLLARY 4.6. Let T, T' be transversals in a measurable foliated chart U which are bijective via the canonical projection map. Then there exists a measurable homotopy $H : T \times \mathbb{R} \rightarrow U$ such that $H(t, 0) = t$ and $H(t, 1) \in T' \cap P_t$ for all $t \in T$, where P_t is the plaque containing t .

DEFINITION 4.7. A chain of charts of a measurable foliated atlas $\mathcal{U} = \{(U_i, \varphi_i)\}_{i \in \mathbb{N}}$ is a finite sequence, $\mathcal{C} = (U_{i_0}, \dots, U_{i_n})$, such that $U_{i_j} \cap U_{i_{j+1}} \neq \emptyset$ for all j . The chain of charts \mathcal{C} covers a path $c : I = [0, 1] \rightarrow X$ if there exists a partition $0 = t_0 < t_1 < \dots < t_n = 1$ of I such that $c([t_{j-1}, t_j]) \subset U_{i_j}$ for all j . The length of a chain of charts $\mathcal{C} = \{U_{i_0}, \dots, U_{i_n}\}$ is n . In a similar way we can define a chain of plaques and a the length of a chain of plaques.

Let $c : I \rightarrow X$ be a foliated path (i.e., a path contained in one leaf). Any chain of charts $\mathcal{C} = (U_{i_0}, \dots, U_{i_n})$ covering c induces a measurable holonomy map $h_{\mathcal{C}}$ between transversals containing $c(0)$ and $c(1)$ like in the topological case.

LEMMA 4.8. Let (X, \mathcal{F}) be a measurable lamination that admits a regular foliated measurable atlas. Let $c : I \rightarrow X$ be a foliated path. Let $\mathcal{C} = (U_{i_0}, \dots, U_{i_n})$ be a chain of charts covering c , and $h_{\mathcal{C}}$ the measurable holonomy map induced by \mathcal{C} . Let T be the domain of $h_{\mathcal{C}}$. Then there exists a measurable homotopy $H : T \times \mathbb{R} \rightarrow X$ such that $H(t, 0) = t$ and $H(t, 1) = h_{\mathcal{C}}(t)$.

PROOF. There exist $n - 1$ transversals S_1, \dots, S_{n-1} such that $S_k \subset U_{i_k} \cap U_{i_{k+1}}$, which meet only plaques that cut these intersections, and only in one point (by Theorem 1.2). By Corollary 4.6, we obtain the required homotopy. \square

LEMMA 4.9. Suppose that \mathcal{F} admits a regular foliated measurable atlas. Let $h : T \rightarrow T'$ be a measurable holonomy map. Then there exists a measurable homotopy $H : T \times \mathbb{R} \rightarrow X$ such that $H(t, 0) = t$ and $H(t, 1) = h(t)$.

PROOF. By Proposition 1.1, we can suppose that T and T' are contained in transversals associated to charts in the regular foliated measurable

atlas. Observe that there exists a countable number of chains of charts covering foliated paths connecting points of T and T' ; the induced measurable holonomy maps are denoted by $\{h_1, \dots, h_n, \dots\}$. The sets

$$B_n = \{t \in T \mid h_n(t) = h(t)\}$$

are measurable. By induction, define

$$C_1 = B_1, \quad C_n = B_n \setminus (C_1 \cup \dots \cup C_{n-1}) \quad (n > 1).$$

These transversals give a partition of T and, by Lemma 4.8, there exist homotopies $H^i : C_i \times \mathbb{R} \rightarrow X$ such that $H^i(t, 0) = t$ and $H^i(t, 1) = h_i(t) = h(t)$. The Borel sets $H^i(C_i \times \{1\})$ form a partition of T' since h is a bijection. Combining these homotopies, we obtain the desired homotopy. \square

PROPOSITION 4.10. *Let U be a tangentially categorical open set, and T a complete transversal. There exists a measurable contraction H of U so that $H(U \times \{1\}) \subset T$.*

PROOF. Let F be a contractible homotopy for U . Therefore $T_F = F(U \times \{1\})$ is a measurable transversal. Since T is a complete transversal, by Proposition 1.1, there exists a countable partition of T_F into measurable transversals $\{T_F^i\}_{i \in \mathbb{N}}$ and injective measurable holonomy maps $h_i : T_F^i \rightarrow T$. By the above arguments, these holonomy maps induce a measurable homotopy $G : T_F \times \mathbb{R} \rightarrow \mathcal{F}$ such that $G(x, 0) = x$ and $G(T_F \times \{1\}) \subset T$. Now, $H = F * G$ is the required contraction. \square

By Proposition 4.3, \mathcal{T}^0 is contained in a categorical open set. Now, we prove an analogous property for each \mathcal{T}^n for $0 < n \leq \dim \mathcal{F}$.

PROPOSITION 4.11. *There exists a measurable triangulation \mathcal{T}' and categorical open sets U^n such that $\mathcal{T}^n \subset U^n$ for $0 < n \leq \dim \mathcal{F}$.*

PROOF. Let $e(x)$ be the n -face (n -simplex without boundary) containing x . Using barycentric division, we can suppose that all n -faces, $0 < n \leq \dim \mathcal{F}$, are contained in an exponential ball centered at the corresponding barycenter (a geodesic ball with radius smaller than the injectivity radius); this triangulation will be called \mathcal{T}' . In fact, we can suppose that the diameter of $e(p)$ is smaller than $i(p)/2$ for any barycenter p . Now, we construct a measurable open set U_n that contains \mathcal{T}^n and such that each of its connected components contains only one n -face and is contained in the respective geodesic ball. This measurable open set contracts to the set of barycenters of \mathcal{T}^n by the exponential map, which completes the proof.

Let \mathcal{B}^n denote the set of barycenters of \mathcal{T}^n . Let $\rho : e(p) \rightarrow \mathbb{R}^+$ be a continuous function and let $N(e(p), \rho)$ denote the neighborhood of $e(p)$ consisting of the union of the balls of radius $\rho(x)$ in the geodesic orthogonal sections of $e(p)$ through x . We define $h^n : \mathcal{T}^n \rightarrow \mathbb{R}$ by $h(x) = d_g(x, \mathcal{T}^n \setminus e(x))$, which is measurable since g and \mathcal{T}' are measurable. Now, let $\rho_p^n : e(p) \rightarrow \mathbb{R}^+$ be given by

$$\rho_p^n(x) = \frac{1}{2} \min\{h(x), i(p)\}.$$

Clearly, $U^n = \bigcup_{p \in \mathcal{B}^n} N(e(p), \rho_p^n)$ is a measurable open set that covers \mathcal{T}^n . Each open set $N(e(p), \rho_p^n)$ is contained in the maximal exponential ball centered at p by definition of ρ_p^n and the conditions satisfied by \mathcal{T}' . These open

sets are disjoint from each other. In fact, if $x \in N(e(p), \rho_p^n) \cap N(e(p'), \rho_{p'}^n)$, then

$$d_g(x, e(p)) = d_g(x, \xi) \leq \frac{1}{2} d_g(x, \mathcal{T}^n \setminus e(p)) \leq \frac{1}{2} d_g(x, e(p')) = \frac{1}{2} d_g(x, \xi')$$

for certain $\xi \in e(p)$ and $\xi' \in e(p')$. By the symmetric argument, $d_g(x, \xi') \leq \frac{1}{2} d_g(x, \xi)$, which is contradiction. Therefore the exponential map defines a measurable contraction of the measurable open set U^n to \mathcal{B}^n . \square

THEOREM 4.12 (Dimensional upper bound). *Let T be a complete transversal for the C^1 measurable lamination $(X, \mathcal{F}, \Lambda)$ with a C^1 measurable triangulation. Then $\text{Cat}(\mathcal{F}, \Lambda) \leq (\dim \mathcal{F} + 1) \cdot \Lambda(T)$.*

PROOF. Measurable laminations of class C^1 admit a C^1 measurable triangulation and a leafwise Riemannian metric [5]. By the Proposition 4.11, there exists a categorical measurable open set U^n containing each set \mathcal{T}^n associated to a measurable triangulation for $0 \leq n \leq \dim \mathcal{F}$. Hence the sets U^n cover X . By Proposition 4.10, $\tau_\Lambda(U^n) \leq \Lambda(T)$ for $0 \leq n \leq \dim \mathcal{F}$. \square

This theorem has important consequences.

COROLLARY 4.13. *Let (X, \mathcal{F}) a minimal C^1 -foliated space. Let Λ be a regular invariant measure of \mathcal{F} without atoms. Then $\text{Cat}_{\text{meas}}(\mathcal{F}, \Lambda) = 0$.*

Recall that a transverse invariant measure of a foliated measurable space is called *ergodic* if it is finite in a complete transversal and any saturated measurable set has null or full measure.

COROLLARY 4.14. *Let $(X, \mathcal{F}, \Lambda)$ be a C^1 -measurable lamination with an ergodic transverse invariant measure without atoms and C^1 -triangulable. Then $\text{Cat}(\mathcal{F}, \Lambda) = 0$.*

COROLLARY 4.15 (Dimensional upper bound of the tangential category). *In these hypothesis, there exists complete transversals with arbitrarily small measure.*

CHAPTER 5

Measurable cohomology

In this chapter, we give a version of the useful cohomological lower bound of the classical LS category, stating that $\text{Nil}(H^*(M, \Gamma)) \leq \text{Cat}(M)$ for any manifold M , where $\text{Nil}(H^*(M, \Gamma))$ denotes the nilpotence order of the cohomology ring $H^*(M, \Gamma)$ with coefficients in any ring Γ [11]. For this purpose, we give an idea of the cohomology of MT-spaces [4, 18, 25].

DEFINITION 5.1 (Measurable prism). A *measurable prism* is a product of a standard Borel space T and a linear region of \mathbb{R}^N (for instance a polygon) with the standard MT-structure. A *measurable simplex* is a measurable prism where the topological fiber is a canonical n -simplex Δ^n . A *measurable singular simplex* on X is an MT-map $\sigma : \Delta^n \times T \rightarrow X$.

Let ω be a usual *singular n -cochain* over a coefficient ring Γ . It is said that ω is *measurable* if $\omega_\sigma : T \rightarrow \Gamma$, $t \mapsto \omega(\sigma|_{\Delta \times \{t\}})$, is measurable for all measurable singular n -simplex σ . The set of measurable cochains is a subcomplex of the complex of usual cochains since the coboundary operator δ preserves the measurability. This measurable subcomplex is denoted by $C_{\text{MT}}^*(X, \Gamma)$, and the coboundary operator restricted to this complex is also denoted by δ .

The *singular measurable cohomology* is defined as usual by

$$H_{\text{MT}}^n(X, \Gamma) = \text{Ker } \delta_n / \text{Im } \delta_{n-1} .$$

The usual cup product gives a well defined exterior product on measurable cochains since the operations in Γ are measurable.

The usual formula $\delta(\omega \smile \theta) = \delta\omega \smile \theta + (-1)^n \omega \smile \delta\theta$ holds. Therefore a cup product is induced in measurable cohomology, obtaining the graded ring $(H_{\text{MT}}^*(X, \Gamma), +, \smile)$.

Any MT-map $f : X \rightarrow Y$ defines a cochain map $f^* : C_{\text{MT}}^*(Y, \Gamma) \rightarrow C_{\text{MT}}^*(X, \Gamma)$ by $f^*(\omega)(\sigma) = \omega(f \circ \sigma)$, which in turn induces a homomorphism between measurable cohomology groups, $f^* : H_{\text{MT}}^*(Y, \Gamma) \rightarrow H_{\text{MT}}^*(X, \Gamma)$.

Let $U \subset X$ be an MT-subspace of X . The inclusion map determines a chain map $i^* : C_{\text{MT}}^*(X, \Gamma) \rightarrow C_{\text{MT}}^*(U, \Gamma)$. The cochain complex $\text{Ker}(i^*)$ will be denoted by $C_{\text{MT}}^*(X, U, \Gamma)$; it consists of the cochains that vanish on any singular simplex contained in U . The corresponding cohomology groups will be called the *measurable relative cohomology groups* of (X, U) . By using the Ker-Coker Lemma like in the classical case, there exists a long exact sequence of cohomology groups (the details are easy to check):

$$\cdots \rightarrow H_{\text{MT}}^n(X, U, \Gamma) \rightarrow H_{\text{MT}}^n(X, \Gamma) \rightarrow H_{\text{MT}}^n(U, \Gamma) \rightarrow H_{\text{MT}}^{n+1}(X, U, \Gamma) \rightarrow \cdots$$

DEFINITION 5.2. Let X, Y be MT-spaces. A *measurable homotopy* or MT-homotopy is an MT-map $H : X \times [0, 1] \rightarrow Y$. It is said that $H(-, 0)$ and $H(-, 1)$ are MT-homotopic maps.

PROPOSITION 5.3 (Invariance by MT-homotopy). *Let $f, g : X \rightarrow Y$ be MT-homotopic maps. Then $f^* = g^* : H_{\text{MT}}^*(Y, \Gamma) \rightarrow H_{\text{MT}}^*(X, \Gamma)$.*

PROOF. The proof is a trivial consequence of the classical proof for singular cohomology. The measurable homotopy induces a chain homotopy between f_* and g_* at the level of the chain complex. The definition is given by cutting the space $\Delta^n \times [0, 1]$ into a finite number of $n + 1$ -prisms $\Pi_i : \Delta^{n+1} \rightarrow \Delta^n \times [0, 1]$. Let H be the measurable homotopy between f and g . The prism operator $P : C^{n+1}(X, \Gamma) \rightarrow C^n(X, \Gamma)$, defined by $P(\omega)(\sigma) = \sum_i I_i \omega(H \circ \sigma|_{\Pi_i})$ [17], where I_i is the orientation factor, is a cochain homotopy between f^* and g^* preserving measurability. Hence f^* and g^* induce the same homomorphism in measurable cohomology. \square

Let \mathcal{U} be a countable measurable open covering of X and let $C_*^{\mathcal{U}}(X, \Gamma)$ the set of chains whose simplices are contained in elements of \mathcal{U} . Clearly, $C_*^{\mathcal{U}}(X, \Gamma)$ is a chain subcomplex of $C_*(X, \Gamma)$. The cochain complex $C_{\mathcal{U}}^*(X, \Gamma)$, dual of $C_*^{\mathcal{U}}(X, \Gamma)$, will be called the cochain complex of X associated to the covering \mathcal{U} . For the measurable setting, it is enough to check the measurability condition for this type of cochains on simplices $\sigma : \Delta \times T \rightarrow X$, with $\sigma(\Delta \times T) \subset U$ for some $U \in \mathcal{U}$. The respective cohomology groups are denoted by $H_{\mathcal{U}, \text{MT}}^*(X, \Gamma)$. Our goal now is to show that these groups are isomorphic to the original ones. The restriction map induces a natural homomorphism $i : H_{\text{MT}}^n(X, \Gamma) \rightarrow H_{\mathcal{U}, \text{MT}}^n(X, \Gamma)$. To prove that i is an isomorphism, we adapt the classical proof by using nice subdivisions of singular simplices adapted to \mathcal{U} in order to give an inverse map preserving measurability.

DEFINITION 5.4. A *linear region* of \mathbb{R}^n is a compact and connected set defined by a finite union of finite intersections of *half-hyperplanes*. Here, a *half-hyperplane* is the set of points $x \in \mathbb{R}^n$ satisfying a linear inequality $\phi(x) \leq c$, where $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ is a non-zero linear map and $c \in \mathbb{R}$. Any linear region has the structure of manifold with corners. A linear region R is *maximal* if its dimension is n . Given a linear region R of \mathbb{R}^n , its boundary ∂R can be expressed as a union of linear regions on affine submanifolds of \mathbb{R}^n of dimension $i < n$; they are called the *i -faces* of the linear region. Two linear regions of \mathbb{R}^n are said to be *attached* if their intersection is a union of faces.

LEMMA 5.5. *A finite union of maximal linear regions of \mathbb{R}^n , $R_1 \cup \dots \cup R_N$, can be expressed as a finite union of maximal linear regions R'_i , $i \in \{1, \dots, N'\}$, such that R'_i and R'_j are attached or disjoint for $i \neq j$, each R'_i is contained in some R_j , and $\bigcup_{i=1}^N \partial R_i = \bigcup_{j=1}^{N'} \partial R'_j$.*

PROOF. Let R_1 and R_2 be two maximal linear regions. The connected components of $R_1 \cap R_2$, $R_1 \setminus R_2$ and $R_2 \setminus R_1$ are linear regions. The maximal linear regions of this kind satisfy the required conditions. Hence the statement is easily proved by induction on N . \square

LEMMA 5.6. *Any finite union of linear regions, such that any pair of them are attached or disjoint, admits a triangulation \mathcal{T} that induces a triangulation on each linear region.*

PROOF. Any linear region can be subdivided by a union of convex linear regions. This subdivision is given by its definition from half-hyperplanes: let $R = \bigcup_{i=1}^K \bigcap_{j=1}^{M_i} H_j^{i,0}$, where the sets $H_j^{i,0}$ are half-hyperplanes, let $H_j^{i,1} = \overline{\mathbb{R}^n \setminus H_j^{i,0}}$. For each i , consider the family

$$D_i = \left\{ A_1 \cap \cdots \cap A_{i-1} \cap \bigcap_{j=1}^{M_i} H_j^{i,0} \cap A_{i+1} \cap \cdots \cap A_K \right\},$$

where $A_k \in \{ \bigcap_{j=1}^{M_k} H_j^{k,i_j} \mid i_j \in \{0,1\} \forall j \}$. The union of the families D_i gives a convex subdivision of R . Hence we can suppose that the linear regions considered are convex. The triangulation is given by standard barycentric subdivision, defining the barycenter of a linear region as its mass center. \square

DEFINITION 5.7 (Subdivision of a measurable simplex). Let $\Delta \times S$ be a measurable n -simplex. A *measurable subdivision* of a measurable simplex is a countable family of measurable n -simplices and MT-embeddings $\phi_i : \Delta \times S_i \rightarrow \Delta \times S$ such that $\phi_i(\Delta \times S_i)$ determines a usual subdivision on each fiber; i.e., the family $\{\phi_i(\Delta \times S_i) \cap (\Delta \times \{s\})\}$ is a usual subdivision of the simplex $\Delta \times \{s\}$ for all $s \in S$.

PROPOSITION 5.8. Let \mathcal{U} be a measurable countable open covering of a measurable simplex $\Delta \times T$. There exists a measurable subdivision of $\Delta \times T$ such that, with notation of Definition 5.7, each $\phi_i(\Delta \times S_i)$ is contained in some element of \mathcal{U} . In fact, there exists a measurable partition $\{T'_j\}_{j \in \mathbb{N}}$ of T such that the subdivision on $\Delta \times T'_j$ is constant; i.e., the same subdivision in each fiber $\Delta \times \{*\}$.

PROOF. Take a base of open subsets of Δ^n given by the barycentric subdivisions, where the simplices of these subdivisions are slightly augmented to be open sets and their closures are also simplices. Order this base of augmented simplices and denote it by $\mathcal{B} = \{\Delta_1, \dots, \Delta_i, \dots\}$. By using Theorem 1.2 on each $U \in \mathcal{U}$, there exists a sequence $\{T_i\}_{i \in \mathbb{N}}$ of Borel subsets of T such that $\Delta \times T = \bigcup_i (\Delta_i \times T_i)$, and each $\overline{\Delta_i} \times T_i$ is contained in some $U \in \mathcal{U}$. Observe that, in general, the family $\{T_i\}_{i \in \mathbb{N}}$ is not a partition of T .

For each $N \in \mathbb{N}$, let $S_{1,\dots,N}^N = \bigcap_{i=1}^N T_i$ and let $I \subset \{1, \dots, N\}$. Define recursively $S_I^N = (\bigcap_{i \in I} T_i) \setminus \bigcup_{J \supsetneq I} \bigcap_{j \in J} T_j$. They are a finite number of disjoint measurable transversals, and, clearly, $\bigcup_{i=1}^N \overline{\Delta_i} \times T_i = \bigsqcup_I (\bigcup_{i \in I} \overline{\Delta_i}) \times S_I^N$.

By compactness, each fiber is contained in a finite union $\bigcup_{i=1}^N (\overline{\Delta_i} \times T_i)$ for some N large enough; in fact, it is contained in some of the sets $(\bigcup_{i \in I} \overline{\Delta_i}) \times S_I^N$. The family of transversals S_I^N is clearly countable, and let $\{S_n\}_{n \in \mathbb{N}}$ be an enumeration of this family. Let $\{S_{n_k}\}_{k \in \mathbb{N}}$ the subfamily of transversals such that their topological fibers $(\bigcup_{i \in I(k)} \overline{\Delta_i})$ for $S_{n_k} = S_{I(k)}^{N(k)}$ cover Δ . Let $T'_1 = S_{n_1}$, and define recursively $T'_k = S'_{n_k} \setminus \bigcup_{j=1}^{k-1} S_{n_j}$. This family is a measurable partition of T and every T'_k is contained in $S_{I(k)}^{N(k)}$. For each $k \in \mathbb{N}$, Lemmas 5.5 and 5.6 provide a triangulation of Δ that

induces subdivisions in the family of simplices $\{\overline{\Delta}_i\}_{i \in I(k)}$. This triangulation is extended to a measurable triangulation in the measurable prism $(\bigcup_{i \in I(k)} \overline{\Delta}_i) \times T'_k$ by taking exactly the same triangulation on each topological fiber. Hence the measurable subdivision induced on $\Delta \times T$ satisfies the conditions of the statement. \square

COROLLARY 5.9. *Let \mathcal{U} be a measurable countable open covering of \mathcal{F} . Then $i : H_{\text{MT}}^n(\mathcal{F}, \Gamma) \rightarrow H_{\mathcal{U}, \text{MT}}^n(\mathcal{F}, \Gamma)$ is an isomorphism for all $n \in \mathbb{N}$.*

PROOF. Notice that any singular chain $\sigma : \Delta \times T^\sigma \rightarrow \mathcal{F}$ induces on $\Delta \times T$ a countable partition of adapted subdivisions with respect to the covering $\sigma^{-1}\mathcal{U}$ by using the above proposition. Let $\{T_i^\sigma \mid i \in \mathbb{N}\}$ denote the partition corresponding to T_σ , and \mathcal{T}_i^σ the subdivision of each fiber of $\Delta \times T_i^\sigma$. For $t \in T_\sigma$, let $i(t)$ be the unique index such that $t \in T_{i(t)}^\sigma$. The expression of the inverse map of i^* on closed cochains is $\rho^*(\omega)(\sigma)(t) = \sum_{\Delta \in \mathcal{T}_{i(t)}^\sigma} \omega(\sigma|_{\Delta \times T_{i(t)}^\sigma})(t)$. \square

Let $C_{\text{MT}}^*(U+V)$ be the cochain complex given by the measurable cochains which vanish on measurable singular simplices that do not lay in either U or V , and let $H_{\text{MT}}^*(U+V)$ denote its cohomology. By the previous corollary, $H_{\text{MT}}^*(\mathcal{F}_{U \cup V})$ is isomorphic to $H_{\text{MT}}^*(U+V)$ via the restriction map. In fact, by using the 5-Lemma, $H_{\text{MT}}^*(\mathcal{F}, \mathcal{F}_{U \cup V})$ is isomorphic to $H_{\text{MT}}^*(\mathcal{F}, U+V)$.

COROLLARY 5.10. *The usual cup product induces a well defined cup product in measurable relative cohomology:*

$$\smile : H_{\text{MT}}^n(\mathcal{F}, \mathcal{F}_U, \Gamma) \times H_{\text{MT}}^m(\mathcal{F}, \mathcal{F}_V, \Gamma) \rightarrow H_{\text{MT}}^{n+m}(\mathcal{F}, \mathcal{F}_{U \cup V}, \Gamma).$$

PROOF. The cup product gives

$$\smile : H_{\text{MT}}^n(\mathcal{F}, \mathcal{F}_U, \Gamma) \times H_{\text{MT}}^m(\mathcal{F}, \mathcal{F}_V, \Gamma) \rightarrow H_{\text{MT}}^{n+m}(\mathcal{F}, U+V, \Gamma).$$

Now, observe that $H_{\text{MT}}^*(\mathcal{F}, U+V, \Gamma)$ is isomorphic to $H_{\text{MT}}^*(\mathcal{F}, \mathcal{F}_{U \cup V}, \Gamma)$. \square

Our goal now is to adapt the long exact sequence of a triple in order to approach some kind of excision result.

A *measurable triple* (X, A, B) is a collection of three MT-spaces such that A is MT-subspace of X and B is MT-subspace of A . Of course, we have the following short exact sequence of complexes:

$$0 \rightarrow C_{\text{MT}}^*(X, A, \Gamma) \rightarrow C_{\text{MT}}^*(X, B, \Gamma) \rightarrow C_{\text{MT}}^*(A, B, \Gamma) \rightarrow 0.$$

The surjectivity of the map to $C_{\text{MT}}^*(A, B, \Gamma)$ can be proved as follows: giving $\omega \in C_{\text{MT}}^*(A, B, \Gamma)$, let $\tilde{\omega}$ be its extension assigning 0 to simplices that are not contained in A ; it is clear that this extension is measurable. Therefore we obtain a short exact sequence in cohomology:

$$\begin{aligned} \cdots \rightarrow H_{\text{MT}}^n(X, A, \Gamma) \rightarrow H_{\text{MT}}^n(X, B, \Gamma) \rightarrow H_{\text{MT}}^n(A, B, \Gamma) \rightarrow \\ H_{\text{MT}}^{n+1}(X, A, \Gamma) \rightarrow \cdots \end{aligned}$$

For the excision statement we refine the conditions on the triple (X, A, B) . Now we require that \overline{B} and $\text{int}(A)$ are measurable sets, and \overline{B} is MT-subspace of $\text{int } A$.

REMARK 8. By a result due to Kallman [22], if a measurable set meets each leaf of a measurable lamination in a σ -compact set, then its closure and interior are measurable. Under these conditions the hypothesis on the triple may be reduced to the usual $A, B \subset X$ and $\overline{B} \subset \text{int}(A)$.

THEOREM 5.11 (Excision for measurable laminations). *Let $\mathcal{U} = \{U, V\}$ be measurable open covering of a measurable lamination \mathcal{F} . Then*

$$H_{\text{MT}}^*(\mathcal{F}, U, \Gamma) \cong H_{\text{MT}}^*(V, U \cap V, \Gamma)$$

via the inclusion map $i : (V, U \cap V) \rightarrow (\mathcal{F}, U)$. Equivalently, if $Z \subset U$ is measurable and closed, then $H_{\text{MT}}^(\mathcal{F}, U, \Gamma) \cong H_{\text{MT}}^*(\mathcal{F} \setminus Z, U \setminus Z, \Gamma)$ via the inclusion.*

The equivalence between the two statements is well known [17].

PROOF. We had seen that the inclusion map $C_{\text{MT}}^*(X, U, \Gamma) \rightarrow C_{\text{MT}}^*(U + V, U, \Gamma)$ induces an isomorphism on measurable cohomology. On the other hand, it is easy to check that the inclusion map $j : C_{\text{MT}}^*(U + V, U, \Gamma) \rightarrow C_{\text{MT}}^*(V, U \cap V, \Gamma)$ also induces an isomorphism in measurable cohomology; in fact, j is an isomorphism of cochain complexes. \square

REMARK 9. It is easy to check that $H_{\text{MT}}^n(T, \Gamma) = 0$ for $n \geq 1$ and any transversal T of a measurable lamination. Hence, by exactness,

$$H_{\text{MT}}^n(\mathcal{F}, T, \Gamma) \cong H_{\text{MT}}^n(\mathcal{F}, \Gamma)$$

for $n \geq 2$. On the other hand, $H_{\text{MT}}^0(\mathcal{F}, \Gamma)$ is the group of measurable maps $f : \mathcal{F} \rightarrow \Gamma$ constant on leaves for any MT-space \mathcal{F} .

REMARK 10. Observe that the exactness of the Mayer-Vietoris sequence holds for measurable cohomology. Let U, V be measurable open sets covering a measurable lamination. The following short sequence is exact:

$$0 \rightarrow C_{\text{MT}}^n(U + V, \Gamma) \rightarrow C_{\text{MT}}^n(U, \Gamma) \oplus C_{\text{MT}}^n(V, \Gamma) \rightarrow C_{\text{MT}}^n(U \cap V, \Gamma) \rightarrow 0.$$

The usual proof of the Mayer-Vietoris principle can be adapted by checking that measurability is always preserved.

Now, we show another description of the singular measurable cohomology for measurable laminations. It will become important to define singular L^2 -cohomology later.

DEFINITION 5.12. Let T be a complete transversal of \mathcal{F} . An *elementary measurable singular n -simplex* relative to T on \mathcal{F} is an MT-map $\sigma : \Delta^n \times T_\sigma \rightarrow \mathcal{F}$ such that $T_\sigma \subset T$ is a Borel subset and $\sigma(\Delta^n \times \{t\}) \subset L_t$ for all $t \in T_\sigma$. Observe that this definition implies that $\sigma^{-1}(L)$ is a countable union of fibers of the measurable simplex for each leaf $L \in \mathcal{F}$. The set of elementary n -simplices is denoted by $EC_n(\mathcal{F})$.

Let $\text{Meas}(T, \Gamma)$ be the group of measurable maps $T \rightarrow \Gamma$. An *elementary n -cochain* over the coefficient ring Γ is a map $\omega : EC_n(\mathcal{F}) \rightarrow \text{Meas}(T, \Gamma)$ such that $\omega(\sigma)$ is supported in T_σ for all $\sigma \in EC_n(\mathcal{F})$, $\omega(\sigma|_{\Delta \times S}) = \omega(\sigma) \cdot \chi_S$ for any measurable $S \subset T_\sigma$ and $\omega(\sigma(\text{id} \times h)) = \omega(\sigma) \circ h$ for all measurable holonomy map $h : A \rightarrow B$ between measurable subsets $A, B \subset T$. The set of elementary n -cochains will be denoted by $EC^n(\mathcal{F}, \Gamma)$, and it is endowed with a group structure induced by Γ .

The coboundary morphism $\delta : EC^{n-1}(\mathcal{F}, \Gamma) \rightarrow EC^n(\mathcal{F}, \Gamma)$ is defined by

$$\delta\omega(\sigma) = \sum_{i=0}^n I_i \omega(\tau_i),$$

where τ_i denote the restriction of σ to each measurable $(n-1)$ -simplex of the boundary and the I_i denote the orientation factors (1 if the orientation of the face agree with the orientation induced by the simplex or -1 otherwise). This is well defined since operations in Γ are measurable. Like in the classical setting, $\delta^2 = 0$ and we have a cochain complex. The *measurable countable-to-one cohomology* groups are defined as usual by $H^n(\mathcal{F}, \Gamma) = \ker \delta_n / \text{Im } \delta_{n-1}$.

Let $f : \mathcal{F} \rightarrow \mathcal{G}$ be an MT-map such that $f^{-1}(L)$ is a countable union of leaves of \mathcal{F} for all $L \in \mathcal{G}$; such a map is said to be *countable-to-one*. By using Lusin's lemma, there exist complete transversals, T of \mathcal{F} and T' of \mathcal{G} , such that $f(T) \subset T'$ and $f : T \rightarrow T'$ is injective. Therefore, measurable countable-to-one cohomology is functorial with respect to countable-to-one maps and complete transversals satisfying the above conditions.

PROPOSITION 5.13. *Elementary cochain complexes relative to different complete transversals are isomorphic. Hence the cohomology groups are independent of the choice of the complete transversal.*

PROOF. We use the notation $EC_n(T)$ for the elementary n -chains relative to a complete transversal T and the notation $EC^n(T)$ for the associated elementary n -cochains. Obviously, it is sufficient to prove that $EC^*(T)$ is isomorphic to $EC^*(T')$ for $T \subset T'$. The inclusion map $i_* : EC_*(T) \hookrightarrow EC_*(T')$ induces a cochain map $i^* : EC^*(T') \rightarrow EC^*(T)$. Now, let us define its inverse. Let $B_1 = T \cap T'$ and $C = T' \setminus T$. By Lusin's lemma (Lemma 1.1), there exists a countable measurable partition $\{B_2, B_3, \dots\}$ of C and measurable holonomy maps $h_i : B_i \rightarrow h_i(B_i) \subset T$ for $i > 1$. For $t \in T$, let $i(t)$ be the unique positive integer such that $t \in B_{i(t)}$. For $\sigma \in EC_n(T')$ and $i \in \mathbb{N}$, define $\sigma_i : \Delta \times h_i(B_i \cap T'_\sigma) \rightarrow \mathcal{F}$ by $\sigma_i(x, t) = \sigma(x, h_i^{-1}(t))$; observe that $\sigma_i \in EC_n(T)$ for all $i \in \mathbb{N}$. Let $\omega \in C^*(T)$, and define $\rho\omega(\sigma)(t) = \omega(\sigma_{i(t)})(h_{i(t)}(t))$, which is an elementary cochain relative to T' . Clearly, $\rho : EC^*(T) \rightarrow EC^*(T')$ is inverse of i^* . \square

COROLLARY 5.14. *Let \mathcal{F} be a one-leaf foliation. Then the standard singular cohomology groups are isomorphic to the measurable countable-to-one ones.*

PROPOSITION 5.15. *Suppose that there exists a countable covering $\mathcal{U} = \{U_n\}_{n \in \mathbb{N}}$ such that all finite intersections are measurably contractible, i.e., there exists a measurable deformation of the inclusion map to a constant map along the leaves. Then the measurable singular cohomology groups and the measurable countable-to-one cohomology ones are isomorphic.*

PROOF. We shall show that this two cohomology groups are isomorphic to a certain notion of measurable Čech cohomology groups related to a nice covering of the measurable lamination, and therefore, they are isomorphic.

The measurable Čech cohomology with respect to a measurable open covering with constant coefficients on a measurable group Γ , is defined as

follows; of course, it is also possible to define it by using sheaves. Let

$$C^s = \prod_{(i_0, \dots, i_s)} F(U_{i_0} \cap \dots \cap U_{i_s}, \Gamma),$$

where $F(U, \Gamma)$ is the set of measurable maps $f : U \rightarrow \Gamma$ constant on the leaves of \mathcal{F}_U . The coboundary map is defined like in the usual Čech complex, and the corresponding cohomology groups are the measurable Čech cohomology groups induced by the covering.

Let $S^r(U, \Gamma)$ (respectively, $S'^r(U, \Gamma)$) denote the set of measurable singular cochains (respectively, countable-to-one) relative to U with coefficients in Γ . Consider the following two double complexes

$$\begin{aligned} C^{r,s} &= \prod_{(i_0, \dots, i_s)} C_{\text{MT}}^r(U_{i_0} \cap \dots \cap U_{i_s}, \Gamma), \\ C'^{r,s} &= \prod_{(i_0, \dots, i_s)} EC^r(U_{i_0} \cap \dots \cap U_{i_s}, \Gamma), \end{aligned}$$

Observe that $C^{*,-1} = C_{\mathcal{U}, \text{MT}}^*(\mathcal{F}, \Gamma)$, and the coboundary map on $C^{*,n}$ is the usual measurable coboundary map. In the same way, $C'^{*,-1} = EC_{\mathcal{U}}^*(\mathcal{F}, \Gamma)$. For the vertical rows, the coboundary map is defined like in the Čech complex. Of course, $C^{-1,*} = \ker(C^{0,s} \rightarrow C^{1,s})$ and $C'^{-1,s} = \ker(EC^{0,s} \rightarrow EC^{1,s})$. An easy computation shows that $C^{-1,*}$ and $C'^{-1,*}$ are the same complex (the measurable Čech complex relative to \mathcal{U}).

For $i, j > -1$, the columns and rows of C and C' have trivial cohomology since the rows are given by measurably contractible spaces and, for the columns, we have the following obvious cochain homotopy to zero. Suppose that \mathcal{U} is well ordered and, for $\sigma : \Delta \rightarrow U_{i_0} \cap \dots \cap U_{i_s}$, define $h\sigma = \sigma : \Delta \rightarrow U_{i_\sigma} \cap U_{i_0} \cap \dots \cap U_{i_s}$, where U_{i_σ} is the first element of \mathcal{U} containing the image of σ . This definition also works for degree -1 , where we consider singular simplices $\sigma : \Delta \rightarrow \mathcal{F}$ whose image is contained in some element of \mathcal{U} . The cochain homotopy $h : C^{r,s} \rightarrow C^{r,s-1}$ is defined by $h\omega(\sigma) = \omega(h\sigma)$. The case of $C'^{r,s}$ is completely analogous.

By a standard argument, the cohomology groups of the first row and first column are isomorphic. Finally the result follows by measurable excision. \square

REMARK 11. The hypothesis is not very restrictive and includes many of the interesting examples. For instance, it holds for measurable suspensions with a good base, measurable simplicial spaces (where the leaves are simplicial complexes) or usual topological foliations.

1. Simplicial, L^r and differentiable measurable cohomology

A measurable lamination may have a measurable simplicial structure. Roughly speaking, it is a simplicial structure on the leaves that varies in a measurable way. It is natural to adapt to these special cases the concept of simplicial and cellular cohomology. Also we introduce the L^2 measurable

cohomology when there exists a transverse invariant measure and the differentiable measurable cohomology for differentiable measurable laminations. Original definitions are given in [4].

DEFINITION 5.16 (Measurable triangulation [4]). A measurable triangulation for a measurable lamination is a measurable family of triangulations $\{\mathcal{T}_L\}_{L \in \mathcal{F}}$. Here, measurability means that the set of their n -simplices are embedded MT-spaces. The image of barycenters of n -simplices, denoted by \mathcal{B}^n , is a transversal. The function $\sigma^n : \Delta^n \times \mathcal{B}^n \rightarrow X$, mapping a barycenter to the embedding $\sigma_p^n : \Delta^n \rightarrow L_p$ given by \mathcal{T}_{L_p} , must be measurable, where Δ^n is the canonical n -simplex. A measurable triangulation is of class C^m if the functions σ_p^n are C^m .

Let \mathcal{T} be a triangulation. An n -cochain over a measurable ring Γ is a measurable map $\omega : \mathcal{B}^n \rightarrow \Gamma$; of course, we identify the barycenters of \mathcal{B}^n with the respective n -simplex. We denote by $C^n(\mathcal{T}, \Gamma)$ the set of simplicial n -cochains; this set is endowed with a ring structure induced by Γ . We define the coboundary operator $\delta : C^n(\mathcal{T}, \Gamma) \rightarrow C^{n+1}(\mathcal{T}, \Gamma)$ as usual by $\delta\omega : \mathcal{B}^{n+1} \rightarrow \Gamma$,

$$\delta\omega(b) = \sum_{\Delta_p^n \subset \partial\Delta_b^{n+1}} b(p) \omega(p)$$

where $b(p)$ is the orientation of the simplex Δ_p^n induced by Δ_b^{n+1} .

Clearly, $\delta^2 = 0$ and we can define the cohomology groups as usual:

$$H^n(\mathcal{T}, \Gamma) = \text{Ker } \delta_n / \text{Im } \delta_{n-1}.$$

PROPOSITION 5.17. *Let (X, \mathcal{F}) be a measurable lamination that admits a measurable triangulation. Then its measurable singular and simplicial cohomology groups are isomorphic.*

PROOF. The standard argument used to prove that measurable singular cohomology is isomorphic to the measurable countable-to-one cohomology (Proposition 5.15) can be adapted easily in order to obtain this result. We only need to check the existence of a nice covering by measurable open sets such that all finite intersections of them are also contractible. The covering will be given by using the star open sets of the 0-simplices. We claim that there exists a measurable countable partition, $\mathcal{T} = \{S_n\}_{n \in \mathbb{N}}$, of \mathcal{B}^0 such that the measurable open sets

$$U_n = \{x \in \mathcal{F} \mid \exists y \in S_n, x \in \text{Star}(y)\}$$

form a nice covering satisfying the above conditions, where $\text{Star}(y)$ denotes the open star set around the 0-simplex y ; i.e., the union

of the interiors of all simplices containing y . Equivalently, we shall prove that any pair of points of each S_n are not connected by any 1-simplex.

Let $\sigma^1 : [0, 1] \times \mathcal{B}^1 \rightarrow \mathcal{F}$ be the 1-simplicial structure of our measurable triangulation. In the measurable prism $[0, 1] \times \mathcal{B}^1$, we have a canonical simplicial structure where the set of 0-simplices is $C^0 = \{0, 1\} \times \mathcal{B}^1$ and the restriction map $\sigma^1 : C^0 \rightarrow \mathcal{B}^0$ has countable fibers. By Lusin's lemma (Proposition 1.1), there exists a measurable partition $\{T_n\}_{n \in \mathbb{N}}$ of C^0 such that $\sigma^1 : T_n \rightarrow \mathcal{B}^0$ is a measurable injection for all $n \in \mathbb{N}$. Each T_n induces

a measurable holonomy map: endow $\{0, 1\}$ with the group structure of \mathbb{Z}_2 and define $h_n : \sigma^1(T_n) \rightarrow \mathcal{B}^0$ by $h_n(\sigma^1(x, t)) = \sigma^1(x + 1, t)$.

Let \mathcal{B}_m^0 be the measurable set of 0-simplices that meet exactly m edges of \mathcal{B}^1 . This family gives a countable partition of \mathcal{B}^0 . Each non-empty intersection $K_{i_1, \dots, i_m}^m = \sigma^1(T_{i_1}) \cap \dots \cap \sigma^1(T_{i_m}) \cap \mathcal{B}_m^0$ defines a domain where only the holonomy maps h_{i_1}, \dots, h_{i_m} are defined. Observe that the sets K_{i_1, \dots, i_m}^m , $m \in \mathbb{N}$, form a measurable countable partition of \mathcal{B}^0 . If $K_{i_1, \dots, i_m}^m \cap h_{i_j}(K_{i_1, \dots, i_m}^m) = \emptyset$ for $1 \leq j \leq m$, then this transversal satisfies our conditions, otherwise we claim that there exists a measurable countable (in fact finite) partition of K_{i_1, \dots, i_m}^m where this is true in each element of the partition. Endow $T_m = \mathcal{B}_m^0 \cap U$ with a Polish topology isomorphic to $[0, 1]$ (the other cases are trivial). For each $x \in K_{i_1, \dots, i_m}^m$, consider a family of open neighborhoods V_n^x such that $\bigcap_n V_n^x = \{x\}$. Hence for n large enough, $V_n^x \cap h_{i_j}(V_n^x) = \emptyset$ for $1 \leq j \leq m$; otherwise there exists an edge connecting x with itself, which contradicts the definition of triangulation. Notice that $h_{i_j}(V_n^x)$ is not open in general, but it is measurable. Now, by compactness, we can choose neighborhoods V_1, \dots, V_N covering K_{i_1, \dots, i_m}^m such that any pair of points on each one of them are not connected by any 1-simplex. Then the desired partition of K_{i_1, \dots, i_m}^m is inductively defined by $B_1^{i_1, \dots, i_m} = V_1$ and $B_k^{i_1, \dots, i_m} = V_k \setminus (\bigcup_{i=1}^{k-1} V_i)$. Finally, the partition \mathcal{T} is given by all the previous partitions $\{B_k^{i_1, \dots, i_m}\}_{k \in \mathbb{N}}^{m \in \mathbb{N}}$. \square

COROLLARY 5.18. *The measurable simplicial cohomology does not depend on the measurable triangulation.*

In the setting of measurable simplicial cohomology we can restrict to a fixed complete transversal parametrizing the measurable simplices. Suppose that there exists a transverse invariant measure Λ and $\Gamma = \mathbb{R}$ or \mathbb{C} (or a measurable subgroup of them). Then, we can work with L^n -measurable cochains, which are equivalence classes of L^n -maps $f : \mathcal{B}^n \rightarrow \Gamma$ that are L^n on each leaf, with the equivalence relation defined by being Λ -almost everywhere equal. The induced cohomology groups are the L^n -measurable cohomology groups associated to the measurable triangulation. For a good definition of the coboundary map, assume that the measurable triangulation is regular; *i.e.*, there is a uniform bound for the number of simplices shearing any face. The invariance of the measure implies that these cohomology groups do not depend on the measurable triangulation. These L^n cohomology groups are not isomorphic in general to the measurable cohomology groups, but they give a simplification of the description of the measurable cohomology groups and help in the search of nonzero cocycles. In the L^2 -case, we can give a structure of Hilbert space to the L_Λ^2 -measurable cochains, and define the Λ -Betti numbers like the Murray-von Neumann dimension of the space of harmonic cochains [9, 4], which is isomorphic to the reduced L^2 -cohomology, defined as the quotient of the kernel of each δ_n over the closure of the image of δ_{n-1} . For measurable laminations with differentiable structure on the leaves, we can define the differentiable measurable cohomology groups [4, 18]. Of course, measurable differentiable classes are measurable sections of $\bigwedge T\mathcal{F}^*$ smooth on the leaves, and the coboundary

operator is the exterior derivative on the leaves; the measurability is preserved by the exterior derivative since partial derivatives are computed by a limit of measurable maps. These cohomologies are related to the leafwise cohomology.

Finally, observe that it is also possible to define a version of the cellular cohomology if we consider a CW-structure on the leaves varying measurably (similarly to a measurable triangulation).

2. Homotopy invariance of the L^2 -cohomology

One of our motivations to define the measurable singular cohomology is its homotopy invariance, which is a simple way to prove that other isomorphic cohomologies are homotopy invariants. The L^2 -cohomology groups are defined with respect to a measurable triangulation. Hence, their homotopy invariance is a good problem. In [18], the proof of this fact is based on the notion of simplicial approximation, without introducing the concept of singular L^2 -cohomology. We solve it by defining a singular version of these groups where the homotopy invariance is easy to prove, and then we show that usual L^2 -cohomology and singular L^2 -cohomology are isomorphic. Of course, we need some conditions on the ambient space of the lamination. We assume the existence of a finite regular foliated atlas such that the transversal associated to each chart has finite Λ -measure, where Λ is the transverse invariant measure. Also, it is supposed that there is a riemannian metric on the leaves that varies measurably on the ambient space.

DEFINITION 5.19. Let T be a complete transversal to \mathcal{F} with finite Λ -measure. A *singular L^2 -cochain* is the equivalence class of an elementary cochain ω such that $\int_{T_\sigma} |\omega(\sigma(-, t))|^2 d\Lambda(t) < \infty$ for all elementary simplex $\sigma : \Delta^n \times T_\sigma \rightarrow \mathcal{F}$ relative to T . The equivalence relation is given by the “ Λ -almost everywhere equal” equivalence relation.

REMARK 12. Notice that L^2 -cochains do not form a Hilbert space. There is no obvious direct way to endow this space with a scalar product.

Of course, the operator δ preserves singular L^2 -cochains and we can define the singular L^2 -cohomology groups $L_\Lambda^2 H_{\text{MT}}^n(\mathcal{F}, \Gamma)$. Let $f : (\mathcal{F}, \Lambda) \rightarrow (\mathcal{G}, \Delta)$ be a countable-to-one MT-map such that $\Lambda(T), \Delta(T') < \infty$ for complete transversals T and T' so that $f : T \rightarrow T'$ is injective. This kind of map is called *comparable* and induces a homomorphism in singular L^2 -cohomology, $f^* : L_\Delta^2 H_{\text{MT}}^n(\mathcal{G}, \Gamma) \rightarrow L_\Lambda^2 H_{\text{MT}}^n(\mathcal{F}, \Gamma)$.

PROPOSITION 5.20. *The singular L^2 -cohomology groups are independent of the choice of the complete transversal.*

PROOF. It can be proved like the analogous statement for countable-to-one cohomology (Proposition 5.13). \square

PROPOSITION 5.21 (MT-homotopy invariance). *Let $f, g : (X, \mathcal{F}, \Lambda) \rightarrow (Y, \mathcal{G}, \Delta)$ be MT-homotopic comparable countable-to-one maps; i.e., there exist a countable-to-one MT-map $H : (X \times \mathbb{R}, \mathcal{F} \times \mathbb{R}, \Lambda) \rightarrow (\mathcal{G}, \Delta)$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$, where $\mathcal{F} \times \mathbb{R}$ denotes the measurable lamination whose leaves are $L \times \mathbb{R}$, $L \in \mathcal{F}$; observe that the homotopy must*

be also comparable by Lusin's lemma. Then f^* and g^* induce the same homomorphism in singular L^2_Λ -cohomology.

PROOF. Looking at the proof of the Proposition 5.3, the fact that the homotopy preserves transversals of finite measure means that the induced cochain homotopy preserves measurable singular L^2 -cochains. Also, observe that the cochain homotopy, P , is well defined at the level of elementary cochains and the homotopy condition, $g^* - f^* = \partial P + P\partial$, holds. \square

PROPOSITION 5.22. *The singular L^2 -cohomology groups are isomorphic to the L^2 - simplicial ones.*

PROOF. It is the same as the proof of Proposition 5.17 with the obvious changes. We only need to give the correct notion of singular L^2 -Čech cohomology. Remember that the chosen nice covering is the covering $\{U_n\}_{n \in \mathbb{N}}$ formed by star open sets around a suitable partition of the 0-simplices (see the proof of Proposition 5.17). The L^2 -Čech cohomology is defined like the measurable one by taking L^2 -maps $f : U_n \rightarrow \Gamma$ constant along the star open sets. \square

QUESTION 5.23. The question about the homotopy invariance of the Λ -Betti numbers was formulated by Connes in [9] and solved by Heitsch and Lazarov in [18]. Could the singular L^2 -cohomology provide a simpler proof of this fact?

3. Computation on examples and applications

EXAMPLE 5.24 (Product and trivial Polish laminations [4]). Let C be a simplicial complex such that each simplex meets only finitely many other simplices. Then the measurable simplicial cohomology of the product MT-space $C \times T$, where T is a standard Borel space, can be identified to the space of measurable maps $f : T \rightarrow H^*(C, \Gamma)$, where $H^*_\Delta(C, \Gamma)$ denotes the usual cohomology of C . A similar result holds for the L^r -cohomology when we consider a measure Λ on T , obtaining $L^r_\Lambda H^*_{\text{MT}}(C \times T, \Gamma) \cong L^r(T, \Gamma; \Lambda) \otimes H^*(C, \Gamma)$.

EXAMPLE 5.25 (Wedges). We compute here the measurable cohomology groups of a measurable wedge. Let \mathcal{F}, \mathcal{G} be measurable laminations, let T be a complete transversal of \mathcal{F} and $\gamma : T \rightarrow \mathcal{G}$ a measurable injection such that $\gamma(T)$ is a complete transversal of \mathcal{G} . Let $\pi : \mathcal{F} \sqcup \mathcal{G} \rightarrow \mathcal{F} \vee \mathcal{G}$ be the projection onto the wedge construction identifying each $t \in T$ with $\gamma(t)$. Suppose that T consists of isolated points, and there are measurable atlases of \mathcal{F} and \mathcal{G} with contractible plaques. Then there exists a measurable open set $U \subset \mathcal{F} \vee \mathcal{G}$ and an MT-homotopy $H : U \times [0, 1] \rightarrow \mathcal{F} \vee \mathcal{G}$ such that $\pi(T) \subset U$, $H(\cdot, 0)$ is the identity map on $\pi(T)$, and $H(\cdot, 1)$ is a retraction of U to $\pi(T)$. Moreover $\pi^{-1}(U) = U' \sqcup U''$, where $T \subset U'$, $\gamma(T) \subset U''$, and H defines MT-homotopies, $H' : U' \times [0, 1] \rightarrow \mathcal{F}$ and $H'' : U'' \times [0, 1] \rightarrow \mathcal{G}$, such that $H'(\cdot, 0)$ and $H''(\cdot, 0)$ are identity maps, and $H'(\cdot, 1)$ and $H''(\cdot, 1)$ are retractions to T and $\gamma(T)$, respectively. By homotopy invariance,

$$C^n(\mathcal{F}, T, \Gamma), \quad C^n(\mathcal{G}, \gamma(T), \Gamma), \quad C^n(\mathcal{F} \vee \mathcal{G}, \pi(T), \Gamma)$$

are respectively isomorphic to

$$C^n(\mathcal{F}, U', \Gamma), \quad C^n(\mathcal{G}, U'', \Gamma), \quad C^n(\mathcal{F} \vee \mathcal{G}, U, \Gamma) .$$

Consider the measurable open coverings $\mathcal{U} = \{\mathcal{F} \setminus T, \mathcal{G} \setminus \gamma(T), U\}$, $\mathcal{U}' = \{\mathcal{F} \setminus T, U'\}$, $\mathcal{U}'' = \{\mathcal{G} \setminus \gamma(T), U''\}$ of $\mathcal{F} \vee \mathcal{G}$, \mathcal{F} and \mathcal{G} , respectively. It is clear that the complexes $C_{\mathcal{U}}^*(\mathcal{F} \vee \mathcal{G}, U, \Gamma)$ and $C_{\mathcal{U}'}^*(\mathcal{F}, U', \Gamma) \oplus C_{\mathcal{U}''}^*(\mathcal{G}, U'', \Gamma)$ are isomorphic. Therefore, by Corollary 5.9, we obtain that

$$H^*(\mathcal{F} \vee \mathcal{G}, \pi(T), \Gamma) \cong H^*(\mathcal{F}, T, \Gamma) \oplus H^*(\mathcal{G}, \gamma(T), \Gamma) .$$

EXAMPLE 5.26. Let $(T^2, \mathcal{F}_\alpha)$ be the K onecker flow, given as a suspension of the rotation $R_\alpha : S^1 \rightarrow S^1$ of $2\pi\alpha$ radians. The case where α is rational is trivial. Thus suppose that α is irrational and let us prove that $H_{\text{MT}}^1(\mathcal{F}_\alpha, \mathbb{Z}_2)$ is non-trivial. The projection of $[0, 1] \times S^1$ to T^2 , given by the suspension, induces a measurable triangulation of \mathcal{F}_α , where the 0-skeleton is the projection of $\{0\} \times S^1$ and the 1-skeleton is the projection of $[0, 1] \times S^1$ (we consider the set of barycenters as the projection of $\{1/2\} \times S^1$). Of course, the 0 and 1 measurable cochains are measurable maps $f : S^1 \rightarrow \mathbb{Z}_2$. We show that the 1-cochain, $\omega \equiv 1 : S^1 \rightarrow \mathbb{Z}_2$, is non-trivial. If ω is trivial, then there exists a measurable map $f : S^1 \rightarrow \mathbb{Z}_2$ and $1 = \omega = f \circ R_\alpha - f$. Of course, it is also true that $1 = \omega \circ R_\alpha = f \circ R_{2\alpha} - f \circ R_\alpha$. Hence $0 = f \circ R_{2\alpha} - f$, showing that f is 2α invariant. By ergodic arguments, f is constant almost everywhere, and therefore $f \circ R_\alpha - f = 0$ almost everywhere, which is a contradiction.

In higher dimension, let $\mathcal{F}_{\alpha_1, \dots, \alpha_n}$ the foliation of T^{n+1} given by the suspension of minimal rotations of S^1 with angles $2\pi\alpha_1, \dots, 2\pi\alpha_n$ radians, where $\alpha_1, \dots, \alpha_n$ are \mathbb{Q} -linear independent. Each leaf of this foliation is an hyperplane dense in T^{n+1} .

Let $[0, 1]^n \subset \mathbb{R}^n$ be the unit cube and let $T = S^1$. The product $[0, 1]^n \times T$ defines a measurable CW-structure on \mathcal{F} given by the suspension projection $p : [0, 1]^n \times T \rightarrow \mathcal{F}$. Let $\omega \equiv 1 : T \rightarrow \mathbb{Z}_2$, which is a CW-cochain of dimension n . Let R_{α_i} be the rotation of S^1 by $2\pi\alpha_i$. If there exists a CW-cochain θ such that $\delta\theta = \omega$, then

$$(3.1) \quad \sum_i (\theta_i \circ R_{\alpha_i} + \theta_i) = 1 ,$$

where $\theta_i = \theta(p|_{e_i \times T})$ and $e_i = \{(x_1, \dots, x_n) \in [0, 1]^n \mid x_i = 0\}$. The proof that ω is non-trivial is more difficult than in the one dimensional case. First of all, any measurable map $f : S^1 \rightarrow \mathbb{Z}_2$ can be considered as a characteristic map $\chi_B : S^1 \rightarrow \mathbb{Z}_2$, where B is a measurable subset on S^1 . Given measurable subsets $B, C \subset S^1$, it is clear that $\chi_B + \chi_C = \chi_{B \triangle C}$, where $B \triangle C = (B \setminus C) \cup (C \setminus B)$. Of course, $\chi_B \circ R_\alpha = \chi_{R_{-\alpha}B}$, hence $\chi_B \circ R_\alpha + \chi_B = \chi_{(R_{-\alpha}B) \triangle B}$. We want to show that, for any family $\{B_1, \dots, B_n\}$ of n measurable subsets of S^1 , such that $\sum_i \chi_{B_i} \circ R_{\alpha_i} + \chi_B = 1$, the set

$$Z = \left\{ z \in S^1 \mid \sum_i \chi_{B_i} \circ R_{\alpha_i} + \chi_B = 0 \right\}$$

has positive measure, which obviously contradicts (3.1). It is not difficult to see this in the case where each B_i is formed by a finite disjoint union of open arcs, and, in fact, the measure of Z depends on the lengths of these arcs. Of

course, the contradiction comes from the existence of integers m, m_1, \dots, m_n such that $2\pi m_1 \alpha_1 + \dots + 2\pi m_n \alpha_n = 2\pi m$.

Now, let $\{B_1, \dots, B_n\}$ be a family of measurable sets and let $\{U_n^i\}_{n \in \mathbb{N}}$ be a family of sequences of open sets in S^1 , where each U_n^i is a finite disjoint union of open arcs such that $B_i \subset U_n^i$ for all n and $\text{length}(U_n^i \setminus B_i) < 2^{-n}$. By induction on the dimension, we can suppose that each B_i has positive measure. Hence the lengths of the arcs U_n^i converge to a positive number. Let

$$Z_n = \left\{ z \in S^1 \mid \sum_i \chi_{U_n^i} \circ R_\alpha + \chi_{U_n^i} = 0 \right\}.$$

By the previous observation, $\liminf_n \text{length}(Z_n) > 0$, and it is easy to see that this limit equals $\text{length}(Z)$.

4. Cohomological lower bound for the measurable LS-category

We had seen that there is a well defined cup product in relative measurable cohomology:

$$\smile : H_{\text{MT}}^n(\mathcal{F}, \mathcal{F}_U, \Gamma) \times H_{\text{MT}}^m(\mathcal{F}, \mathcal{F}_V, \Gamma) \rightarrow H_{\text{MT}}^{n+m}(\mathcal{F}, \mathcal{F}_{U \cup V}, \Gamma).$$

As a consequence of the cohomology concepts developed for measurable laminations we obtain the following corollary.

COROLLARY 5.27. *Let \mathcal{F} be a measurable lamination, and let $H^*(\mathcal{F}, \Gamma)$ the measurable cohomological ring of \mathcal{F} over a measurable ring Γ . Then*

$$\text{Nil}(H^*(\mathcal{F}, \Gamma)) \leq \text{Cat } \mathcal{F}.$$

PROOF. Let $\{U_1, \dots, U_N\}$ be a covering by tangentially contractible measurable open sets. Of course, the map $H_{\text{MT}}^*(\mathcal{F}, \mathcal{F}_{U_i}, \Gamma) \rightarrow H_{\text{MT}}^*(\mathcal{F}, \Gamma)$ of the cohomological exact sequence is onto since each U_i is categorical. Let x_1, \dots, x_N be cohomology classes in $H_{\text{MT}}^*(\mathcal{F})$ and take a preimage ω_i of each x_i in $H_{\text{MT}}^*(\mathcal{F}, \mathcal{F}_{U_i}, \Gamma)$. Therefore $\omega_1 \smile \dots \smile \omega_N \in H_{\text{MT}}^*(\mathcal{F}, \mathcal{F}, \Gamma) = 0$, and this product projects to $x_1 \smile \dots \smile x_N$ in $H_{\text{MT}}^*(\mathcal{F}, \Gamma)$. \square

EXAMPLE 5.28. Let $(T^2, \mathcal{F}_\alpha)$ be the K onecker flow, considered as a suspension of the rotation $R_\alpha : S^1 \rightarrow S^1$ of $2\pi\alpha$ radians. The case where α is rational is trivial (it is a foliation by compact leaves). Then suppose that α is irrational, and let us show that $H^1(\mathcal{F}_\alpha, \mathbb{Z}_2)$ is non trivial. The projection of $[0, 1] \times S^1$ to T^2 , given by the suspension of R_α , induces a measurable triangulation of \mathcal{F}_α , where the 0-skeleton is the projection of $\{0\} \times S^1$ and the 1-skeleton is the projection of $[0, 1] \times S^1$; the set of barycenters is the projection of $\{1/2\} \times S^1$. Of course, measurable cochains of degrees 0 and 1 are measurable maps $f : S^1 \rightarrow \mathbb{Z}_2$. In [25], it is shown that the 1-cochain $\omega = 1 : S^1 \rightarrow \mathbb{Z}_2$, is non-trivial. By Corollary 5.27 and the dimensional bound, it follows that $\text{Cat}(\mathcal{F}_\alpha) = 2$.

In higher dimension, we consider the foliation $\mathcal{F}_{\alpha_1, \dots, \alpha_n}$ of T^{n+1} given by the suspension of rotations $R_{\alpha_1}, \dots, R_{\alpha_n}$ of S^1 , where $\alpha_1, \dots, \alpha_n$ are \mathbb{Q} -linear independent. Each leaf of $\mathcal{F}_{\alpha_1, \dots, \alpha_n}$ is a hyperplane dense in T^{n+1} . Let $[0, 1]^n \subset \mathbb{R}^n$ be the unit cube and let $T \equiv S^1$. The product $[0, 1]^n \times T$ gives a measurable CW-structure on \mathcal{F} given by the projection $p : [0, 1]^n \times T \rightarrow \mathcal{F}$ of

the suspension. Let $\omega = 1 : T \rightarrow \mathbb{Z}_2$, which is a CW-cochain of dimension n . Let τ_i the CW-cochain of dimension 1 satisfying $\tau_i(p|_{[0,1]_j \times T}) = \delta_{ij}$, where

$$[0, 1]_j = \{(x_1, \dots, x_n) \in [0, 1]^n \mid x_i = 0 \text{ for } i \neq j\}.$$

Clearly $\omega = \tau_1 \smile \dots \smile \tau_n$. With similar arguments, we can see that ω is non trivial by Example 5.26. Then Corollary 5.27 and the dimensional upper bound yields $\text{Cat}(\mathcal{F}_{\alpha_1, \dots, \alpha_n}) = n + 1$.

REMARK 13. We do not discuss here about the possibility of a similar lower bound for the Λ -category. In many general situations, this invariant is zero and a lower bound is not so interesting. In the cases where there exists a complete transversal T meeting each leaf at exactly one point, the number $\int_T \text{Nil } H^*(L_t, \Gamma) d\Lambda(t)$ is well defined and gives a lower bound for the Λ -category. Of course, the fact that the Λ -category is a trivial invariant in many interesting cases is bad news, but this will be used in the last part of this work to produce a non-trivial secondary invariant of this kind.

CHAPTER 6

Critical points

As suggested by the classical theory, because of the possible applications, it is interesting to consider Hilbert laminations to study the relation between our measurable versions of the tangential category and the critical points of a smooth function.

Recall the following terminology. A *Hilbert manifold* is a separable, Hausdorff space endowed with an atlas where the charts are homeomorphisms to open subsets of separable Hilbert spaces. A C^2 function on a complete C^2 Hilbert manifold, $f : M \rightarrow \mathbb{R}$, is called *Palais-Smale* whenever, for any sequence (p_n) in M , if $(f(p_n))$ is bounded and $(df(p_n))$ converges to zero, then (p_n) contains a convergent subsequence. If M is compact (in the finite dimensional case), then all differentiable maps are Palais-Smale. Let $\text{Crit}(f)$ be the set of critical points of f . In this section, we adapt a theorem due to J. Schwartz [31], which states that, for a bounded from below Palais-Smale function on a C^2 Hilbert manifold, $f : M \rightarrow \mathbb{R}$, we have $\text{Cat}(M) \leq \# \text{Crit}(f)$.

For the rest of the section, we consider C^2 measurable Hilbert laminations. Their definition is analogous to the definition of measurable lamination: the leafwise topological model is now open balls in a separable Hilbert space instead of \mathbb{R}^n , and the tangential part of the changes of coordinates are C^2 maps between open subsets of Hilbert spaces. Each leaf is a Hilbert manifold, which is endowed with a Riemannian metric that varies in a measurable way in the ambient space. Observe that the main difference with the finite dimensional case is that the leaves may not be locally compact. We also suppose that the leafwise Riemannian metric is complete.

To define the Λ -category of a measurable Hilbert lamination with a transverse invariant measure Λ , we consider only contractible measurable open sets. The reason is that Proposition 3.5 seems to be difficult to generalize to this infinite dimensional setting, as well as other details about measurability. However, Theorem 2.10 holds for measurable Hilbert laminations too, and therefore we can use the coherent extension $\tilde{\Lambda}$ of Λ .

Notice that the differential map of a function varies in a measurable way in the ambient space, since its definition is a limit of measurable maps. The set of C^2 MT-maps will be noted by $C^2(\mathcal{F})$. For a function $f \in C^2(\mathcal{F})$, we set $\text{Crit}_{\mathcal{F}}(f) = \bigcup_{L \in \mathcal{F}} \text{Crit}(f|_L)$.

The cotangent bundle $T\mathcal{F}^*$ is a measurable vector bundle, whose zero section $\theta : X \rightarrow T\mathcal{F}^*$ is measurable with measurable image. For any $f \in C^2(\mathcal{F})$, its differential map $df : X \rightarrow T\mathcal{F}^*$ is measurable, and we have $\text{Crit}_{\mathcal{F}}(f) = df^{-1}(\theta(X))$. Thus $\text{Crit}_{\mathcal{F}}(f)$ is measurable.

Recall that a C^1 isotopy on a C^1 -Hilbert manifold M is a differentiable map $\phi : M \times \mathbb{R} \rightarrow M$ such that $\phi_t = \phi(\cdot, t) : M \rightarrow M$ is a diffeomorphism $\forall t \in [0, 1]$ and $\phi_0 = \text{id}_M$.

DEFINITION 6.1 (Measurable tangential isotopy). Let (X, \mathcal{F}) be a C^1 measurable Hilbert lamination. A *measurable tangential isotopy* on (X, \mathcal{F}) is a C^1 map $\phi : X \times \mathbb{R} \rightarrow X$ such that the functions $\phi_t : X \rightarrow X$ are MT-diffeomorphisms $\forall t$ with $\phi_0 = \text{id}_X$, and the maps $\phi^x : \mathbb{R} \rightarrow X$, with $\phi^x(t) = \phi_t(x)$, are differentiable $\forall x$ $\phi : \mathcal{F} \times \mathbb{R} \rightarrow \mathcal{F}$ is C^1 , where $\mathcal{F} \times \mathbb{R}$ is the C^1 measurable lamination in $X \times \mathbb{R}$ with leaves of the form $L \times \mathbb{R}$, $L \in \mathcal{F}$.

EXAMPLE 6.2 (Construction of a measurable tangential isotopy [27]). A tangential isotopy can be constructed on a Hilbert manifold by using a C^1 tangent vector field V . There exists a flow $\phi_t(p)$ such that $\phi_0(p) = p$, $\phi_{t+s}(p) = \phi_t(\phi_s(p))$ and $d\phi_t(p)/dt = V(\phi_t(p))$. From the way of obtaining ϕ [27, 10], it follows that the same kind of construction for a measurable C^1 tangent vector field on a measurable Hilbert lamination (X, \mathcal{F}) induces a measurable isotopy on (X, \mathcal{F}) .

Now we obtain a measurable isotopy from the gradient flow. It will be modified by a control function α in order to have some control in the deformations induced by the corresponding isotopy. Let ∇f be the gradient tangent vector field of f ; i.e., the unique tangent vector field satisfying $df(v) = \langle v, \nabla f \rangle$ for all $v \in T\mathcal{F}$. Take the C^1 vector field $V = -\alpha(|\nabla f|) \nabla f$, where $\alpha : [0, \infty) \rightarrow \mathbb{R}^+$ is C^∞ , $\alpha(t) \equiv 1$ for $0 \leq t \leq 1$, $t^2\alpha(t)$ is monotone non-decreasing and $t^2\alpha(t) = 2$ for $t \geq 2$. The flow $\phi_t(p)$ of V is defined for $-\infty < t < \infty$ [31], and it is called the *modified gradient flow*.

Let us define a partial order relation “ \ll ” for the critical points of f . First, we say that $x < y$ if there exists a regular point p such that $x \in \alpha(p)$ and $y \in \omega(p)$, where $\alpha(p)$ and $\omega(p)$ are the α - and ω -limits of p . Then $x \ll y$ if there exists a finite sequence of critical points, x_1, \dots, x_n , such that $x < x_1 < \dots < x_n < y$.

We use the notation $\gamma(x)$ for the ϕ -orbit of a point x .

LEMMA 6.3. *Let $T \subset X$ be an isolated transversal. Then there exists a measurable open set U containing T such that $\text{Cat}(U, \mathcal{F}, \Lambda) \leq \Lambda(T)$. We can suppose also that U is tangentially categorical contracting to T .*

PROOF. The tubular neighborhood of T contracts to T (see Lemma 4.8 and observe that its proof generalizes to measurable Hilbert laminations). Hence its relative category is smaller than $\Lambda(T)$. \square

In the following proposition, $\{W_1, W_2, \dots\}$ denotes a foliated measurable atlas. Let $\text{Crit}_{\mathcal{F}}^\infty(f)$ be the union of plaques that contain infinite critical points of f . Observe that this set is a measurable open set.

PROPOSITION 6.4. *If $\tilde{\Lambda}(\text{Crit}_{\mathcal{F}}^\infty(f)) > 0$, then $\tilde{\Lambda}(\text{Crit}_{\mathcal{F}}(f)) = \infty$.*

PROOF. For each chart W_i , let $\pi_i : W_i \rightarrow T_i$ be the transversal projection. Since $\tilde{\Lambda}(\text{Crit}_{\mathcal{F}}^\infty(f)) > 0$, we have $\Lambda(\pi_i(\text{Crit}_{\mathcal{F}}^\infty(f))) > 0$ for some $i \in \mathbb{N}$.

Therefore

$$\begin{aligned} \tilde{\Lambda}(\text{Crit}_{\mathcal{F}}^{\infty}(f)) &\geq \int_{\pi_i(\text{Crit}_{\mathcal{F}}^{\infty}(f))} \#(\text{Crit}_{\mathcal{F}}(f) \cap \pi_i^{-1}(t)) d\Lambda(t) \\ &= \infty \cdot \Lambda(\pi_i(\text{Crit}_{\mathcal{F}}^{\infty}(f))) = \infty. \quad \square \end{aligned}$$

REMARK 14. The set $\text{Crit}_{\mathcal{F}}^{\infty}(f)$ contains all non-isolated critical points of $\text{Crit}_{\mathcal{F}}(f)$. If $\tilde{\Lambda}(\text{Crit}_{\mathcal{F}}(f)) < \infty$, then the saturation of $\text{Crit}_{\mathcal{F}}^{\infty}(f)$ is a null-transverse set. Hence we can restrict our study to the case where all critical points are isolated.

The definition of a Palais-Smale condition is needed for a version of the Lusternik-Schnirelmann Theorem on Hilbert manifolds. For measurable Hilbert laminations, a possible definition could be to take functions that satisfies Palais-Smale condition on all (or almost all) leaves. This is very restrictive: it would mean that the set of relative minima meets each leaf in a relatively compact set (which is non-empty when f bounded from below), and therefore would exist a complete transversal meeting each leaf at one point. We shall use the following more general condition.

DEFINITION 6.5. A measurable ω -Palais-Smale (or ω -PS) function is a function $f \in C^2(\mathcal{F})$ such that any ϕ -orbit have non empty ω -limit and, for any $p \in \text{Crit}_{\mathcal{F}}(f)$, the set $\{x \in \text{Crit}_{\mathcal{F}}(f) \mid p \ll x\}$ is compact, and this set is empty if and only if p is a relative minimum. A measurable α -Palais-Smale (or α -PS) function is defined analogously by considering the set $\{x \in \text{Crit}_{\mathcal{F}}(f) \mid x \ll p\}$.

Of course, f is ω -PS if and only if $-f$ is α -PS.

LEMMA 6.6. Suppose that $\text{Crit}_{\mathcal{F}}(f)$ is an isolated transversal. The modified gradient flow ϕ (see Example 6.2) satisfies the following properties:

- (i) The flow runs towards lower level sets of f , i.e., $f(p) \geq f(\phi_t(p))$ for $t > 0$.
- (ii) The invariant points of the flow are just the critical points of f .
- (iii) A point is critical if and only if $f(\phi_t(p)) = f(p)$ for some $t \neq 0$.
- (iv) The points in α - and ω -limits are critical points if they are non empty.

PROOF. We work with the leaf topology. Therefore these results can be proved in each leaf, considered as a C^2 Hilbert manifold. These properties follows from the work of J. Schwartz [31].

Under these conditions, the α - and ω -limits are connected sets that consist of critical points if they are non-empty. If $\omega(p)$ is infinite, then all of its points are non-isolated, contradicting the assumption. \square

DEFINITION 6.7 (Critical sets). For a measurable ω -Palais-Smale function, the set of minima is non-empty in any leaf. Let p be a critical point. By the properties of the flow ϕ , either p is a relative minimum, or there exists another critical point x such that $p < x$. Define M_0, M_1, \dots inductively by

$$\begin{aligned} M_0 &= \{x \in \text{Crit}_{\mathcal{F}}(f) \mid \nexists y \text{ such that } x \ll y\}, \\ M_i &= \{x \in \text{Crit}_{\mathcal{F}}(f) \mid \forall y \ x \ll y \Rightarrow y \in M_0 \cup \dots \cup M_{i-1}\}. \end{aligned}$$

Clearly, M_0 contains all relative minima on the leaves. We also set $C_0(f) = M_0$ and $C_i(f) = M_i \setminus (M_0 \cup \dots \cup M_{i-1})$. Observe that, if $x \in \omega(p)$ and $x \in C_i(f)$ for some i , then $\omega(p) \subset C_i(f)$; there is an analogous property for the α -limit. The notation C_i will be used if there is no confusion and they will be called the *critical sets* of f . Let $p \ll p^*$. Then $i_{p^*} < i_p$, where i_p and i_{p^*} are the indexes such that $p \in C_{i_p}$ and $p^* \in C_{i_{p^*}}$.

The set of relative minima of a bounded from below measurable ω -PS function is always non-empty in any leaf.

THEOREM 6.8. *Let (X, \mathcal{F}) be a measurable Hilbert lamination endowed with a measurable Riemannian metric on the leaves, and let f be a measurable ω -PS map. Suppose that $\text{Crit}_{\mathcal{F}}(f)$ is an isolated transversal. Then $\text{Cat}(\mathcal{F}) \leq \#\{\text{critical sets of } f\}$.*

THEOREM 6.9. *Let $(X, \mathcal{F}, \Lambda)$ be a measurable Hilbert manifold endowed with a measurable Riemannian metric on the leaves and with a transverse invariant measure, and let f be a measurable ω -PS map. Then $\text{Cat}(\mathcal{F}, \Lambda) \leq \tilde{\Lambda}(\text{Crit}_{\mathcal{F}}(f))$.*

REMARK 15. Notice also that the first Theorem 6.8 gives a slight sharpening of the classical theorem of Lusternik-Schnirelmann, since the number of critical sets may be finite even when the number of all critical points are infinite. We see this in the following example. Let $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = \sin(x)$, which is not an P-S function in the classical sense, but it satisfies our measurable Palais-Smale condition in the one leaf lamination \mathbb{R} . An easy computation shows that there are two critical sets, the set of relative minima and the set of relative maxima, yielding the inequality $\text{Cat}(\mathbb{R}) \leq 2$. On more complicated examples, this improved version could be used to find better upper bounds of the classical LS category.

REMARK 16. From the existence of a measurable Riemannian metric, two disjoint isolated transversals, can be separated by measurable open sets. In fact, we can suppose that the closures of connected components of these measurable open sets contain only one point of these transversals (see Proposition 4.3).

LEMMA 6.10. *Let (X, \mathcal{F}) be a measurable Hilbert lamination and let $f_n : (X, \mathcal{F}) \rightarrow (X, \mathcal{F})$ be a sequence of MT-maps. Suppose that $(f_n(x))$ converges for all $x \in X$. Then $\lim_n f_n$ is measurable.*

PROOF. Measurable open sets generate the σ -algebra of (X, \mathcal{F}) , in fact, by Theorem 1.2 the measurable foliated charts are a generating set also. Then it is enough to prove that $(\lim_n f_n)^{-1}(V)$ is measurable for any foliated chart V . For each $V \equiv B \times T$ there exists a sequence of measurable closed sets $\{F_n\}_{n \in \mathbb{N}}$ such that $V = \bigcup_n F_n$. For instance, if B is an open ball we can take $F_n \equiv \overline{B_n} \times T$, where B_n are open balls of smaller radius than B but converging to it. Now, it is clear that

$$\begin{aligned} (\lim_n f_n)^{-1}(V) &= \{x \in X \mid \exists N \text{ such that } f_n(x) \in F_N \ \forall n \geq N\} \\ &= \bigcup_{N=1}^{\infty} \bigcap_{n \geq N} f_n^{-1}(F_N), \end{aligned}$$

which is, clearly, a measurable set. \square

REMARK 17. It is known, in basic measure theory, that the limit of real measurable functions is also measurable. Lemma 6.10 is not a direct consequence from this fact because the measurable structure is not the Borel σ -algebra corresponding to the topology.

COROLLARY 6.11. *The family $\{C_i\}_{i \in \mathbb{N} \cup \{0\}}$ is a measurable partition of $\text{Crit}_{\mathcal{F}}(f)$.*

PROOF. Clearly, this family is a partition by the properties of ϕ . To show that each C_i is measurable, observe that $\text{Crit}_{\mathcal{F}}(f)$ is a transversal, and $X \setminus \text{Crit}_{\mathcal{F}}(f)$ is measurable with open intersection with each leaf. Clearly, the α - and ω -limit functions, $\alpha, \omega : X \rightarrow \text{Crit}_{\mathcal{F}}(f)$, are measurable by Lemma 6.10 since $\alpha = \lim_n \phi_{-n}$ and $\omega = \lim_n \phi_n$. Observe that $\alpha(X \setminus \text{Crit}_{\mathcal{F}}(f)) = \text{Crit}_{\mathcal{F}}(f) \setminus C_0$. Therefore C_0 is measurable. By Remark 16, there exists a measurable open set U_0 containing C_0 and separating it from $\text{Crit}_{\mathcal{F}}(f) \setminus C_0$; in fact, each connected component contains only one point of C_0 . Take the measurable open set $O_0 = \bigcup_{n \in \mathbb{N}} \phi_{-n}(U_0)$. It is easy to see that

$$\alpha(X \setminus (O_0 \cup \text{Crit}_{\mathcal{F}}(f))) = \text{Crit}_{\mathcal{F}}(f) \setminus (C_0 \cup C_1).$$

Therefore C_1 is a measurable set. By a recursive argument, we obtain that all sets C_i are measurable. \square

LEMMA 6.12. *Let (X, \mathcal{F}) be a measurable Hilbert lamination and let (Y, \mathcal{G}) be a finite dimensional lamination such that $Y \subset X$ is a measurable open set and the inclusion map is an MT-embedding considering in Y . Then there exists a countable family of measurable foliated charts of (X, \mathcal{F}) , $\{U_n \equiv B_n \times T_n\}_{n \in \mathbb{N}}$, covering Y and such that each fiber $B_n \times \{t\}$ is a foliated chart (in a topological sense) of \mathcal{G} in a leaf of \mathcal{F}_Y .*

PROOF. Let U be a measurable foliated chart of \mathcal{F} . Since Y is open in X , $U \cap Y$ is also measurable and open. Hence a measurable foliated atlas induces a covering of Y by measurable open sets and a measurable foliated atlas of (Y, \mathcal{F}_Y) . Since \mathcal{G} is a sublamination of \mathcal{F}_Y , by Theorem 1.2, we can choose a measurable foliated atlas such that the plaques are products of the form $B^k \times B'$ where B^k are open balls in \mathbb{R}^k where $\dim \mathcal{G} = k$ and B' is a ball in the orthogonal complement of \mathcal{G} in a leaf of \mathcal{F} centered in the origin. Of course the projection $\pi : (B^k \times B) \times T \rightarrow (B^k \times \{0\}) \times T$ defines an MT-map that works like a tubular neighborhood of a plaque of a family of leaves of \mathcal{G} . Of course $B^k \times (B' \times T)$ is a measurable chart for \mathcal{G} , hence the family of tubular neighborhoods is a measurable atlas of \mathcal{G} and \mathcal{F}_Y simultaneously. \square

PROOF OF THEOREM 6.8. By Remark 16 and Lemma 6.3, there exists a disjoint family of measurable open sets, $\{U_i\}$ ($i \in \mathbb{N} \cup \{0\}$), such that each U_i contains and contracts to C_i , and $\text{Cat}(U_i, \mathcal{F}, \Lambda) \leq \Lambda(C_i)$. We also assume that $\bar{U}_i \subset \tilde{U}_i$, where \tilde{U}_i is another categorical measurable open set that contracts to C_i .

Let $U'_0 = \bigcup_{n \in \mathbb{N}} \phi_{-n}(U_0)$. This set is open since $C_0(f)$ consists of relative minima, and contracts to $C_0(f)$ by using the MT isotopy ϕ . The set $X_1 =$

$X \setminus U'_0$ is a measurable closed set, and $C_1(f)$ is the set of relative minima of the restriction $f|_{X_1}$. The set X_1 consists of the critical points that do not belong to $C_0(f)$ and the regular points connecting these critical points according to the relation “ \ll ”. Let $F_1 = U_1 \cap X_1$ and $F'_1 = \bigcup_{n \in \mathbb{N}} \phi_{-n}(F_1)$. The set F'_1 is open in X_1 and closed in X . Let us prove that F'_1 is contained in a measurable open set U'_1 such that there exists a measurable deformation H with $H(U'_1 \times \{1\}) \subset \widetilde{U}_1$.

Of course, $(X \setminus \text{Crit}_{\mathcal{F}}(f), \phi)$ is a measurable lamination of dimension 1, where the leaves are the flow lines of ϕ . These flow lines are embedded submanifolds and they admit a countable covering by measurable foliated charts in the sense of Lemma 6.12. Therefore there exists a measurable atlas of $(X \setminus \text{Crit}_{\mathcal{F}}(f), \phi)$, $\{(W_n, \varphi_n)\}_{n \in \mathbb{N}}$, with $\varphi_n : W_n \rightarrow B^1 \times B_n \times T_n$, where B^1 is an open interval in \mathbb{R} , B_n is an open ball centered at the origin in a separable Hilbert space, and T_n is a standard space. We can suppose also that this measurable atlas is locally finite and let $\pi_n : W_n \rightarrow \varphi_n^{-1}(B^1 \times \{0\} \times T_n)$ be given by the canonical projection $B^1 \times B_n \times T_n \rightarrow B^1 \times \{0\} \times T_n$.

By similar arguments, we can suppose that $F'_1 \subset \bigcup_n W_n \subset \bigcup_{i \in \mathbb{N}} \phi_{-i}(U_1)$, $\varphi_n(B^1 \times \{0\} \times T_n) \subset F'_1$ and $\bigcup_n W_n$ is a *semisaturated* set; i.e., if $x \in \bigcup_n W_n$ then $\phi_t(x) \in \bigcup_n W_n$ for all $t \in [0, \infty)$. Let $\{\lambda_n\}$ be a measurable partition of unity subordinated to $\{W_n\}$ [5] such that each λ_n is continuous on $\varphi_n^{-1}(B^1 \times B_n \times \{z\})$ for all $z \in T_n$. For each $x \in \bigcup_n W_n$, let $I(x) \subset \mathbb{N}$ be the set of numbers n such that the semiorbit $\phi_{[0, \infty)}(x)$ meets W_n . The isotopy $\phi_t|_{F'_1}$ contracts F'_1 to C_1 . We extend the deformation $\phi_t|_{F'_1 \setminus C_1(f)}$ to the neighborhood $\bigcup_n W_n$. This extension can be defined as follows: for $x \in \bigcup_n W_n$, $t \in \mathbb{R}$ and $n \in I(x)$, there is a unique positive real number $r(x, t, n)$ such that $\phi_{r(x, t, n)}(x) = \gamma(x) \cap \pi_n^{-1}(\phi_t(\pi_n(x)))$. Let $H : V_1 \times \mathbb{R} \rightarrow X$ be the continuous map defined by $H(x, t) = \phi_{s(x, t)}(x)$, where

$$s(x, t) = \sum_{k \in I(x)} \lambda_k(x) r(x, t, k).$$

For $x \in \bigcup_n W_n$ and $t \in \mathbb{R}$, there exists $k_1, k_0 \in I(x)$ such that $r(x, t, k_1) \leq s(x, t) \leq r(x, t, k_0)$. It is clear that there exists $\lim_{t \rightarrow \infty} \phi_{r(x, t, n)}(x) \in \overline{U}_1 \subset \widetilde{U}_1$. Let $p \in C_1$ and let $x \in F'_1 \setminus C_1$ with $\omega(x) = p$. Since $\bigcup_n W_n$ is semisaturated and it is contained in $\bigcup_{i \in \mathbb{N}} \phi_{-i}(U_1)$, for all $r(x, t, k_1) < t' < r(x, t, k_0)$, $\phi_{t'}(x) \in \widetilde{U}_1$ for t large enough. Therefore $\lim_{t \rightarrow \infty} H(x, t) \in \widetilde{U}_1$ for all $x \in \bigcup_n V_n$. Then the measurable open subset $V'_1 = \bigcup_n V_n$ is \mathcal{F} -categorical (by a standard change of parameter). Finally, if \widetilde{U}_1 is small enough, $U'_1 = V'_1 \cup \widetilde{U}_1$ is \mathcal{F} -categorical by a telescopic argument [17] and $F'_1 \subset U'_1$.

This process can be continued inductively by taking $X_n = X \setminus (U'_0 \cup \bigcup_{i=1}^{n-1} F'_i)$ and using the same trick to define U'_n , observing that $C_n(f)$ is the set of relative minima of $f|_{X_n}$. \square

PROOF OF THEOREM 6.9. By Remark 14, we can restrict the study to the case where $\text{Crit}_{\mathcal{F}}(f)$ is an isolated transversal. The previous proof also shows that $\text{Cat}(\mathcal{F}, \Lambda)$ is a lower bound for the sum of the measures of the critical sets. Since the critical sets form a partition of $\text{Crit}_{\mathcal{F}}(f)$, the proof is complete. \square

Part 2

Topological category

CHAPTER 7

LS category on usual laminations

In this chapter we deal with the question of define the measurable versions of LS category to usual laminations. The tangential version is well known [7, 8, 33], now we work with usual open sets and not measurable sets open in the leaf topology. This point of view can applied to the Λ -category providing a new invariant of topological laminations. We shall adapt many of the properties of measurable categories and find other new interesting qualities.

1. Definition and first properties of the topological Λ -category

We refer to [6] for the basic notion and definitions about laminations, foliated chart, foliated atlas, holonomy pseudogroup and transverse invariant measure. They are recalled here in order to fix notations.

Let X be a Polish space. A *foliated chart* in X is a pair (U, φ) such that U is an open subset of X and $\varphi : U \rightarrow B^n \times S$ is an homeomorphism, where B^n is an open ball of \mathbb{R}^n and S is a Polish space. The sets $B^n \times \{*\}$ are called the *plaques* of the chart, and the sets of the form $\varphi^{-1}(\{*\} \times S)$ are called the *associated transversals*. The map $U \rightarrow S$ is called the projection associated to (U, φ) . A *foliated atlas* is a family of foliated charts, $\{(U_i, \varphi_i)\}_{i \in I}$, that covers X and the change of coordinates between the charts preserves the plaques; i.e., they are locally of the form $\varphi_i \circ \varphi_j^{-1}(x, s) = (f_{ij}(x, s), g_{ij}(s))$; these maps g_{ij} form the holonomy cocycle associated to the foliated atlas. A *lamination* \mathcal{F} on X is a maximal foliated atlas satisfying the above hypothesis. The plaques of the foliated charts of a maximal foliated atlas form a base of a finer topology of X , called the *leaf topology*. The connected components of the leaf topology are called the *leaves* of the foliation. The *dimension* of the lamination is the dimension of the plaques when all of them are open sets of the same Euclidean space.

A foliated atlas, $\mathcal{U} = \{(U_i, \varphi_i)\}_{i \in \mathbb{N}}$, is called *regular* if it satisfies the following properties:

- (a) It is locally finite.
- (b) If a plaque P of any foliated chart (U_i, φ_i) meets another foliated chart (U_j, φ_j) , then $P \cap U_j$ is contained in only one plaque of (U_j, φ_j) .
- (c) If $U_i \cap U_j \neq \emptyset$, then there exists a foliated chart (V, ψ) such that $U_i \cup U_j \subset V$, $\varphi_i = \psi|_{U_i}$ and $\varphi_j = \psi|_{U_j}$.

Any topological lamination admits a regular foliated atlas, also we can assume that all the charts are locally compact. For a regular foliated atlas

$\mathcal{U} = \{(U_i, \varphi_i)\}_{i \in \mathbb{N}}$ with $\varphi_i : U_i \rightarrow B_{i,n} \times S_i$, the maps g_{ij} generate a pseudogroup on $\bigsqcup_i S_i$. Holonomy pseudogroups defined by different foliated atlases are equivalent in the sense of [16], and the corresponding equivalence class is called the *holonomy pseudogroup* of the lamination; it contains all the information about its transverse dynamics. A *transversal* T is a topological set of X so that, for every foliated chart, (U, φ) , the corresponding projection restricts to a local homeomorphism $U \cap T \rightarrow S$. A transversal is said to be *complete* if it meets every leaf. On any complete transversal, there is a representative of the holonomy pseudogroup which is equivalent to the representative defined by any foliated atlas via the projection maps defined by its charts. A *transverse invariant measure* is a measure on the space of any representative of the holonomy pseudogroup invariant by its local transformations; in particular, it can be given as a measure on a complete transversal invariant by the corresponding representative of the holonomy pseudogroup.

We saw that usual laminations induce measurable laminations (see Example

ref:functor O) and therefore we can pass some terms of measurable laminations to usual laminations like measurable transversals. Any transverse invariant measure can be extended to all measurable transversals so that they become invariant by measurable transformations that keep each point in the same leaf [9].

Observe that a measure is regular if the measure of a set B is the infimum of the measures of the open sets containing it and the supremum of the measures of compact sets contained in B . Note that measures in the conditions of the Riesz representation theorem are regular, so it is not a very restrictive condition on measures.

REMARK 18. Observe that the tangential model of the charts could be changed in order to give a more general notion of lamination: instead of taking open balls of \mathbb{R}^n as B^n , we could take connected and locally contractible Polish spaces or separable Hilbert spaces. Also, it is possible to define the notion of C^r foliated structure by assuming that the tangential part of changes of coordinates are C^r , with the leafwise derivatives of order $\leq r$ depending continuously on the transverse coordinates. We can speak about regular atlases in Hilbert laminations but we cannot assume that its foliated charts are locally compact.

Let us recall the definition of tangential category [7, 8]. A lamination (X, \mathcal{F}) induces a foliated measurable structure \mathcal{F}_U in each open set U . The space $U \times \mathbb{R}$ admits an obvious foliated structure $\mathcal{F}_{U \times \mathbb{R}}$ whose leaves are products of leaves of \mathcal{F}_U and \mathbb{R} . Let (Y, \mathcal{G}) be another measurable lamination. A foliated map $H : \mathcal{F}_{U \times \mathbb{R}} \rightarrow \mathcal{G}$ is called a *tangential homotopy*, and it is said that the maps $H(\cdot, 0)$ and $H(\cdot, 1)$ are *tangentially homotopic*. We use the term *tangential deformation* when $\mathcal{G} = \mathcal{F}$ and $H(-, 0)$ is the inclusion map of U . A deformation such that $H(-, 1)$ is constant on the leaves of \mathcal{F}_U is called a *tangential contraction* or an *\mathcal{F} -contraction*; in this case, U is called a *tangentially categorical* or *\mathcal{F} -categorical* open set. The *tangential category* is the lowest number of categorical open sets that cover the measurable lamination. On one leaf foliations, this definition agrees with

the classical category. The category of \mathcal{F} is denoted by $\text{Cat}(\mathcal{F})$. It is clear that it is a tangential homotopy invariant.

Now, we introduce the relative category that will be useful for further applications.

DEFINITION 7.1. Let $U \subset X$ be an open subset. The *relative category* of U , $\text{Cat}(U, \mathcal{F})$, is the minimum number of \mathcal{F} -categorical open sets in X that cover U .

REMARK 19. Clearly, $\text{Cat}(U, \mathcal{F}) \leq \text{Cat}(\mathcal{F}_U)$.

PROPOSITION 7.2 (Subadditivity of the relative category). *Let $\{U_i\}_{i \in \mathbb{N}}$ be a countable family of open subsets of X . Then*

$$\text{Cat}\left(\bigcup_i U_i, \mathcal{F}\right) \leq \sum_i \text{Cat}(U_i, \mathcal{F}) .$$

The following result is about the structure of a tangentially categorical open set.

LEMMA 7.3 (Singhof-Vogt [33]). *Let \mathcal{F} be a foliation of dimension m and codimension n on a manifold M , let U be an \mathcal{F} -categorical open set, let $x \in U$, let $D \subset U$ be a transverse manifold of dimension n , and suppose that x belongs to the interior of D . Then there exists a neighborhood E of x in D such that any leaf of \mathcal{F}_U meets E in at most one point.*

Observe that this lemma extends to the case of Hilbert laminations with the same proof. Hence, if the ambient space is separable, the final step of a tangential contraction is a countable union of local transversals of the foliation. Therefore it can be measured by a transverse invariant measure.

Let Λ be a transverse invariant measure for \mathcal{F} and let U be an open set. Define

$$\tau_\Lambda(U) = \inf\{\tilde{\Lambda}(H(U \times \{1\})) \mid H \text{ is a tangential deformation of } U\} .$$

Then the Λ -category of (\mathcal{F}, Λ) is defined as

$$\text{Cat}(\mathcal{F}, \Lambda) = \inf_{\mathcal{U}} \sum_{U \in \mathcal{U}} \tau_\Lambda(U) ,$$

where \mathcal{U} runs in the open coverings of X . If the homotopies used in this definition are required to be C^r on leaves with the leafwise derivatives of order $\leq r$ depending continuously on the transverse coordinates, then the term C^r Λ -category is used, with the notation $\text{Cat}^r(\mathcal{F}, \Lambda)$. Observe that, for one leaf laminations with the transverse invariant measure given by the counting measure, the measured category agrees with the classical LS category.

PROPOSITION 7.4. *If Λ is positive in non-empty open transversals, then $\tau_\Lambda(U) < \infty \Rightarrow U$ is \mathcal{F} -categorical.*

PROOF. We prove that if $H : U \times I \rightarrow X$ is a non-leaf constant tangential homotopy, $\tilde{\Lambda}(H(U \times \{1\})) = \infty$.

By taking a small foliated chart around a point in a leaf of \mathcal{F}_U where H_1 non-leaf constant, we can suppose that $U \approx \mathbb{R}^n \times T$ is a distinguished open set where $T \subset X$ is an embedded transversal, $H(T \times \{1\})$ is an embedded

transversal homeomorphic to T and $H(U \times \{1\}) \subset U_\alpha$, where U_α is a foliated chart. Also, we suppose that $H(-, 1)$ induce an injective map $\overline{H}_1 : T \rightarrow T_\alpha$ (or $\pi : H(T \times \{1\}) \rightarrow T_\alpha$ injective, with π a local quotient map), where T_α is an associated transversal to U_α .

Hence, there exist x, y in a plaque of U such that $H(x, 1)$ and $H(y, 1) \in U_\alpha$ and $H(x, 1) \neq H(y, 1)$, setting $x \in T$. Also, since \overline{H}_1 is injective, the image by $H(-, 1)$ of this plaque is contained in only one plaque of U_α , and a path $H(\sigma \times \{1\})$, where σ is a foliated path connecting x and y in the corresponding plaque of U . In this way, if P' notes the plaque of U_α containing $H(P \times \{1\})$, then

$$\begin{aligned} \int_{P'} \mathcal{X}_{H_1(T \times D) \cap P'} d\nu &= \#(H_1(T \times D) \cap P') \\ &\geq \#(H_1(P)) \\ &\geq \#(H_1(\sigma)) \\ &= \infty. \end{aligned}$$

We are going to prove that there exists an open neighborhood V of $H(x, 1)$ in $H(T \times \{1\})$, homeomorphic, via π , with a open subset of T_α , that satisfies $\#(H(((T \times D) \cap P_s) \times \{1\})) = \infty$ for all $s \in V$, where P_s denotes the plaque meeting s . If this property is false, then:

For all neighborhood V there exists $s_V \in V$ and a plaque P of U that contracts, via H , to a finite amount of points contained in the plaque P_{s_V} ; one of these points is s_V . But plaques are connected, then P_{s_V} contracts, via H , to s_V .

Now, let W_1 and W_2 be disjoint open sets in U_α containing $H(x, 1)$ and $H(y, 1)$ respectively. The inverse images, via $H(-, 1)$, are disjoint open sets that contain x and y respectively. Their projections in T give us open sets in T with non-empty intersection since x and y belongs to the same plaque; this intersection will be denoted by W . Since $H(-, 1)|_T$ is a homeomorphism, $H(W \times \{1\})$ is an open neighborhood of $H(x, 1)$ in $H(T \times \{1\})$, hence there exists $s_{H(W \times \{1\})}$ giving by the above property. Let P^W the unique plaque in U that contracts, via H , to $s_{H(W \times \{1\})}$. This plaque is unique since \overline{H}_1 is injective, hence meets W and therefore cuts $H(-, 1)^{-1}(W_1)$ and $H(-, 1)^{-1}|_{T \times D}(W_2)$. But this implies that $s_{H(W \times \{1\})} \in W_1 \cap W_2$, a contradiction since W_1 and W_2 are disjoint.

Now, we obtain that

$$\begin{aligned} \tilde{\Lambda}(H_1(U)) &= \tilde{\Lambda}(H_1(\mathbb{R}^n \times T)) \\ &\geq \int_V \left(\int_P \#(H_1(\mathbb{R}^n \times T) \cap P) d\nu \right) d\Lambda(P) \\ &= \infty \cdot \Lambda(V) \\ &= \infty. \end{aligned}$$

Latest equality uses the fact that Λ is positive in non-empty open sets. \square

The first property of the Λ -category that we prove is its homotopy invariance. A tangential homotopy equivalence h from \mathcal{F} to \mathcal{G} induces a

canonical bijection between the sets of transverse invariant measures on \mathcal{G} and \mathcal{F} , which holds since h induces an equivalence between the holonomy pseudogroups [2]. In fact, these properties were proved for measurable laminations and the key fact is to use Lemma 3.3.

PROPOSITION 7.5 (The Λ -category is a tangential homotopy invariant). *Let $(X, \mathcal{F}, \Lambda)$ and (Y, \mathcal{G}, Δ) be tangentially homotopy equivalent laminations with transverse invariant measures. Then $\text{Cat}(\mathcal{F}, \Lambda) = \text{Cat}(\mathcal{G}, \Delta)$.*

PROOF. Let $h : (\mathcal{F}, \Lambda) \rightarrow (\mathcal{G}, \Delta)$ be a tangential homotopy equivalence, and let g be a homotopy inverse of h . Let $\{U_n\}$ ($n \in \mathbb{N}$) be a covering of Y by open sets. Then $\{h^{-1}(U_n)\}$ is a covering of X by open sets. We will prove that $\tau_\Lambda(h^{-1}(U_n)) \leq \tau_\Delta(U_n)$ for all $n \in \mathbb{N}$. Let H^n be a tangential contraction on each U_n , and let F be a tangential homotopy connecting the identity map and $g \circ h$. Let

$$G^n : h^{-1}(U_n) \times \mathbb{R} \xrightarrow{h \times \text{id}} U_n \times \mathbb{R} \xrightarrow{H^n} X \xrightarrow{g} Y.$$

Then $K^n : h^{-1}(U_n) \times \mathbb{R} \rightarrow X$, defined by

$$K(x, t) = \begin{cases} F(x, 2t) & \text{if } t \leq 1/2 \\ G^n(x, 2t - 1) & \text{if } t \geq 1/2, \end{cases}$$

is a tangential contraction for all n . Lemma 3.3 yields

$$\Lambda(K(h^{-1}(U) \times \{1\})) = \Lambda(g(H(U \times \{1\}))) \leq \Delta(H(U \times \{1\})).$$

Hence $\tau_\Lambda(h^{-1}(U_n)) \leq \tau_\Delta(U_n)$ for all $n \in \mathbb{N}$. Therefore $\text{Cat}(\mathcal{F}, \Lambda) \leq \text{Cat}(\mathcal{G}, \Delta)$. The inverse inequality is analogous. \square

Of course, the above proposition has an obvious C^r version.

DEFINITION 7.6. Let $(X, \mathcal{F}, \Lambda)$ be a lamination with a transverse invariant measure. A *null-transverse* set is a measurable set B such that $\widetilde{\Lambda}(B) = 0$.

The following propositions are elementary.

PROPOSITION 7.7. *Let $(X, \mathcal{F}, \Lambda)$ be a lamination with a regular transverse invariant measure, and let B be a null-transverse set. Then $\text{Cat}(\mathcal{F}, \Lambda)$ can be computed by using only coverings of $X \setminus \text{sat}(B)$ by open sets in X , where $\text{sat}(B)$ denotes the saturation of B in \mathcal{F} . If B is saturated and closed, then $\text{Cat}(\mathcal{F}, \Lambda) = \text{Cat}(\mathcal{F}_{X \setminus B}, \Lambda_{X \setminus B})$.*

PROPOSITION 7.8. *Let T be a transversal such that $\#T \cap L \leq \text{Cat}(L)$ for all $L \in \mathcal{F}$. Then $\Lambda(T) \leq \text{Cat}(\mathcal{F}, \Lambda)$.*

PROOF. Let $\{U_n\}_{n \in \mathbb{N}}$ be a tangential open covering of \mathcal{F} and let H^n be tangential contractions for each U_n . Hence $T^n = H^n(U \times \{1\})$ are transverse sets and $\bigcup_n T^n$ meets all leaves. Therefore $\Lambda(T) \leq \text{Cat}(\mathcal{F}, \Lambda)$ since $U_n \cap L$ is categorical in L for all $n \in \mathbb{N}$ which $U_n \cap L \neq \emptyset$ (see e.g. [33]), and Λ is holonomy invariant. \square

Let M be a manifold, S a locally compact Polish space, $h : \pi_1(M) \rightarrow \text{Homeo}(S)$ a homomorphism, and Λ a measure on S invariant by $h(\pi_1(M))$. Then the quotient space, $\widetilde{M} \times_h S$, of $\widetilde{M} \times S$, by the relation $(x, s) \sim$

$(g^{-1}x, h(g)(s))$, $g \in \pi_1(M)$ is called the suspension of the homomorphism h , where \tilde{M} is the universal covering space of M . The suspension has a structure of lamination whose leaves are covering spaces of M , and Λ is a transverse invariant measure.

PROPOSITION 7.9. *If \mathcal{F} is a lamination induced by a suspension, then $\text{Cat}(\mathcal{F}, \Lambda) \leq \text{Cat}(M) \cdot \Lambda(S)$.*

PROPOSITION 7.10. *For a manifold M and a locally compact Polish space T , let $M \times T$ be foliated as a product. Then $\text{Cat}(M \times T, \Lambda) = \text{Cat}(M) \cdot \Lambda(T)$ for every measure Λ on T , considered as a transverse invariant measure of $M \times T$ with the lamination with leaves $M \times \{*\}$.*

PROPOSITION 7.11. *Let $\{U_n\}$ ($n \in \mathbb{N}$) be a covering by saturated open sets of $(X, \mathcal{F}, \Lambda)$. Then $\text{Cat}(\mathcal{F}, \Lambda) \leq \sum_n \text{Cat}(\mathcal{F}_{U_n}, \Lambda_{U_n})$. Here, the equality holds if $\{U_n\}$ is a partition.*

DEFINITION 7.12. Let (\mathcal{F}, Λ) be a lamination with transverse invariant measure, the *relative Λ -category* of U is defined by

$$\text{Cat}(U, \mathcal{F}, \Lambda) = \inf_{\mathcal{U}} \sum_{V \in \mathcal{U}} \tau_{\Lambda}(V),$$

where \mathcal{U} runs in the family of countable open coverings of U .

REMARK 20. Observe that $\tau_{\Lambda}(V)$ is defined by using tangential homotopies deforming V in the ambient space. Clearly, $\text{Cat}(U, \mathcal{F}, \Lambda) \leq \text{Cat}(\mathcal{F}_U, \Lambda)$.

PROPOSITION 7.13 (Subadditivity of relative Λ -category). *Let $\{U_i\}$ ($i \in \mathbb{N}$) be a countable family of open subsets of X . Then $\text{Cat}(\bigcup_i U_i, \mathcal{F}, \Lambda) \leq \sum_i \text{Cat}(U_i, \mathcal{F}, \Lambda)$.*

Now, we explain an important lemma that allows us to cut tangentially categorical open sets into small ones, in order to compute the Λ -category when the measure is finite on compact sets. In this process, a null transverse set is generated, but we know that this kind of set can be removed in the computation of Λ -category.

LEMMA 7.14. *Let (\mathcal{F}, Λ) be a lamination with a transverse invariant measure that is finite on compact sets. Let $h : S \rightarrow T'$ be a holonomy map defined on a locally compact domain that can be extended to a neighborhood W of \bar{S} . Then there exists an open neighborhood T of \bar{S} where h is defined and such that $\Lambda(\partial T) = 0$.*

PROOF. Let d be a metric on W that induces its topology. For each $\epsilon > 0$, let

$$S^{\epsilon} = \{x \in T \mid d(x, S) < \epsilon\}.$$

There exists $\delta > 0$ such that $S^{\delta} \subset W$. The boundaries ∂S^{ϵ} are disjoint. If all of them have nontrivial Λ -measure, then

$$\Lambda(S^{\delta}) \geq \sup \left\{ \sum_{\epsilon \in I} \Lambda(\partial S^{\epsilon}) \mid I \subset (0, \delta), I \text{ finite} \right\} = \sum_{0 < \epsilon < \delta} \Lambda(\partial S^{\epsilon}) = \infty,$$

where the last equality follows from the fact that any non-countable sum of positive numbers is infinite. But $\Lambda(\bar{S})$ is finite since Λ is finite on compact sets. \square

LEMMA 7.15. *Let (\mathcal{F}, Λ) be a lamination with a transverse invariant measure that is finite on compact sets. Let U be a tangentially categorical open set and let T be a complete transversal. Then there exist a partition $\{F, U_n\}_{n \in \mathbb{N}}$ of U such that F is a closed and null transverse set, each U_n is open, and there exists a tangential homotopy $H : \bigcup_n U_n \times \mathbb{R} \rightarrow \mathcal{F}$ such that $H(\bigcup_n U_n \times \{1\}) \subset T$.*

PROOF. Let G be a tangential contraction of U . By Lemma 7.3, $H(U \times \{1\}) = S$ is a countable union of transversals. Therefore there exist a countable covering of S , $S = \bigcup_i S_i$, by transversals such that, for each $i \in \mathbb{N}$, there exists a holonomy map h_i with domain S_i and image contained in T (since T is a complete transversal). Each h_i induce a tangential deformation $H^i : S_i \times \mathbb{R} \rightarrow \mathcal{F}$ such that $H^i(S_i \times \{1\}) = h_i(S_i)$. By Lemma 7.14, we can suppose that $\Lambda(\partial S_i) = 0$ for $i \in \mathbb{N}$.

Now, take $T_1 = S_1$ and define recursively $T_n = \text{int}(V_n \setminus \bigcup_{i=1}^{n-1} S_i)$ and $K = U \setminus \bigcup_n T_n$. Of course, K is closed in S and it is null transverse since $K \subset \bigcup_i \partial S_i$. Finally, we obtain the partition by taking $U_n = H(-, 1)^{-1}(T_n)$ and $F = H(-, 0)^{-1}(K)$. The tangential contraction H for $\bigcup_n U_n$ is defined as follows:

$$H(x, t) = \begin{cases} G(x, 2t) & \text{if } t \leq \frac{1}{2} \\ H^i(G(x, 1), 2t - 1) & \text{if } t \geq \frac{1}{2} \text{ and } x \in U_i. \quad \square \end{cases}$$

The way to define deformations used in the above prove will be useful in other sections of this work. This idea is specified by defining the following homotopy operation.

DEFINITION 7.16. Let $H : X \times \mathbb{R} \rightarrow Y$ and $G : Z \times \mathbb{R} \rightarrow Y$ be tangential deformations (hence it is supposed that $X, Z \subset Y$, and $H(-, 0)$ and $G(-, 0)$ are inclusion maps) such that $H(X \times \{1\}) \subset Z$. Then let $H * G$ be the tangential deformation of X defined by:

$$H * G(x, t) = \begin{cases} H(x, 2t) & \text{if } t \leq \frac{1}{2} \\ G(H(x, 1), 2t - 1) & \text{if } t \geq \frac{1}{2}. \end{cases}$$

2. Λ -category of compact-Hausdorff laminations

In this section, we compute the Λ -category of a lamination (X, \mathcal{F}) with all leaves compact and Hausdorff leaf space. Suppose that Λ is finite on compact sets.

In this setting, there is a nice description for the local dynamics [14]. For a leaf L , there exists a (topological) transversal such that there is an open foliated embedding of a suspension $i_L : \tilde{L} \times_{h_L} U_L \rightarrow (X, \mathcal{F})$, where $h_L : \pi_1(L) \times U_L \rightarrow U_L$ is an action defining the holonomy of L , with finite orbits and Hausdorff orbit space. A transversal satisfying these conditions for some leaf will be called a *slice*. We omit i_L in the notations for the sake of simplicity. The set of saturations of slices is a base of the saturated topology where the open sets of this topology is the saturation of open sets.

For compact laminations, it is also true that a transverse invariant measure induces a measure in the leaf space, which is defined in the following way. For each measurable subset $B \subset X/\mathcal{F}$, there exists a transverse set

$T \subset X$ such that $\pi(T) = B$ and $\pi_T : T \rightarrow B$ is a Borel isomorphism, where $\pi : X \rightarrow X/\mathcal{F}$ is the canonical projection. Let $\Lambda_{\mathcal{F}}(B) = \Lambda(T)$, which does not depend on the choice of such T [12, 35, 26], and defines a measure $\Lambda_{\mathcal{F}}$ on X/\mathcal{F} .

REMARK 21. If Λ is finite on compact sets, then, for any σ -compact set K in a transversal, we have

$$\Lambda(K) = \inf\{\Lambda(V) \mid V \text{ is open, } K \subset V\}.$$

This means that Λ is *externally regular* on σ -compact sets.

PROPOSITION 7.17. *For any leaf L , there exists a slice U_L such that $\Lambda(\partial U_L) = 0$.*

PROOF. Consequence of Lemma 7.14 and the fact that, according to notation of Lemma 7.14, there exists an slice U_L for each $L \in \mathcal{F}$ and $\delta > 0$ (depending on L) such that U_L^ε is an slice for $0 < \varepsilon < \delta$. \square

THEOREM 7.18. *Let $(X, \mathcal{F}, \Lambda)$ be a lamination with all leaves compact, Hausdorff leaf space, and a transverse invariant measure that is finite on compact sets. Then*

$$\text{Cat}(\mathcal{F}, \Lambda) = \int_{X/\mathcal{F}} \text{Cat}(L) d\Lambda_{\mathcal{F}}(L).$$

PROOF. Let $\{B_0 = X, \dots, B_\alpha, B_{\alpha+1}, \dots\}$ be the Epstein filtration of (X, \mathcal{F}) [15]; it is defined by transfinite induction: $B_0 \setminus B_1$ is the set of leaves with trivial holonomy, which is open and dense in X , and B_α is the set of leaves with non-trivial holonomy in $\bigcap_{\beta < \alpha} B_\beta$. This filtration is always countable, even though it may involve infinite ordinals α . We obtain a partition of X given by the sets $F_\alpha = B_\alpha \setminus B_{\alpha+1}$, which are laminations with all leaves compact, trivial holonomy and dense in B_α . Forgetting the correspondence between the inclusion relation on the family $\{B_\alpha\}$ and the order of the indices α , we can change the indices of the family $\{F_\alpha\}$ to denote it by $\{F_i\}_{i \in \mathbb{N}}$. Given any $\varepsilon > 0$, for each leaf $L \subset F_i$, choose a slice U_L^i satisfying the following properties:

- (i) $\overline{U_L^i}$ is compact;
- (ii) $\Lambda(\partial U_L^i) = 0$;
- (iii) $\Lambda((\bigcup_{L \in F_i/\mathcal{F}} U_L^i) \setminus F_i) < \varepsilon/2^i$;
- (iv) U_L^i meets each leaf of F_i in at most one point. The leaves of F_i meeting U_L^i are homeomorphic to each other.

Property (i) is given by the local compactness. Properties (ii) and (iii) are consequences of the external regularity on σ -compact sets, and property (iv) follows from the continuity of every volume map on F_i (see [14] for details). By the Lindelöf property, there exist leaves $L_1^i, \dots, L_n^i, \dots$ in F_i so that the family $\{L_n^i \times_{h_{L_n^i}} U_{L_n^i}^i\}_{i,n \in \mathbb{N}}$ covers X . By induction on n , define $A_1^i = L_1^i \times_{h_{L_1^i}} U_{L_1^i}^i$ and

$$A_n^i = (L_n^i \times_{h_{L_n^i}} U_{L_n^i}^i) \setminus (\overline{A_1^i} \cup \dots \cup \overline{A_{n-1}^i}), \quad n > 1.$$

The families $\{A_n^i\}_{n \in \mathbb{N}}$ consist of disjoint open sets that cover F_i up to the saturation of a null-transverse set. Each A_n^i is a suspension with base L_n^i

and transverse fiber $S_n^i = U_{L_n^i} \setminus (\overline{A_1^i} \cup \dots \cup \overline{A_{n-1}^i})$. From these properties, it follows that

- (i) $\Lambda(\bigcup_{n=1}^{\infty} S_n^i \cap F_i) = \Lambda_{\mathcal{F}}(F_i/\mathcal{F})$,
- (ii) $\Lambda((\bigcup_{n=1}^{\infty} S_n^i) \setminus F_i) < \varepsilon/2^i$.

Then, by Proposition 7.9 and using the dimensional upper bound of the LS category of manifolds, we get

$$\begin{aligned}
 \text{Cat}(\mathcal{F}, \Lambda) &\leq \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \text{Cat}(L_n^i) \cdot \Lambda(S_n^i) \\
 &= \sum_{i=1}^{\infty} \int_{\bigcup_n S_n^i} \sum_{n=1}^{\infty} \text{Cat}(L_n^i) \chi_{S_n^i} d\Lambda \\
 &= \sum_{i=1}^{\infty} \int_{\bigcup_n S_n^i \cap F_i} \sum_{n=1}^{\infty} \text{Cat}(L_n^i) \chi_{S_n^i} d\Lambda \\
 &\quad + \sum_{i=1}^{\infty} \int_{\bigcup_n S_n^i \setminus F_i} \sum_{n=1}^{\infty} \text{Cat}(L_n^i) \chi_{S_n^i} d\Lambda \\
 &\leq \sum_{i=1}^{\infty} \left(\int_{F_i/\mathcal{F}} \text{Cat}(L) d\Lambda_{\mathcal{F}}(L) \right. \\
 &\quad \left. + (\dim \mathcal{F} + 1) \cdot \Lambda \left(\left(\bigcup_{n=1}^{\infty} S_n^i \right) \setminus F_i \right) \right) \\
 &< \int_{X/\mathcal{F}} \text{Cat}(L) d\Lambda_{\mathcal{F}}(L) + (\dim \mathcal{F} + 1) \cdot \varepsilon. \quad \square
 \end{aligned}$$

REMARK 22. In usual laminations there exist two notions of tangential category and Λ -category, since we can compute the measurable invariants in the induced measurable laminations. In order to distinguish these invariants we shall use the notations $\text{Cat}_{\text{Top}}(\mathcal{F})$, $\text{Cat}_{\text{Meas}}(\mathcal{F})$, $\text{Cat}_{\text{Top}}(\mathcal{F}, \Lambda)$ and $\text{Cat}_{\text{Meas}}(\mathcal{F}, \Lambda)$ if there is possibility of confusion.

Finally, we show that the measurable and topological Λ -categories do not agree if the transverse measure does not satisfy good properties. We consider the subspace

$$S = \{ (x, y) \mid x^2 + y^2 = 1 \text{ or } x^2 + y^2 = 1 - 1/n, n \in \mathbb{N}, n > 1 \} \subset \mathbb{R}^2.$$

Let S_l be the circle of radius l . On $T = S \setminus S_1$, define a homeomorphism $f : T \rightarrow T$ so that $f|_{S_{1-\frac{1}{n}}} : S_{1-\frac{1}{n}} \rightarrow S_{1-\frac{1}{n}}$ is the rotation with angle $\frac{2\pi}{n}$. On $S^1 \times S$, define a foliated space whose leaves on $S^1 \times T$ are the leaves of the suspension $\mathbb{R} \times_f T$, and whose leaves on $S^1 \times S_1$ are the fibers $S^1 \times \{*\}$. The Epstein filtration is given by $B_0 = X$ and $B_1 = S^1 \times S_1$. A transverse invariant measure is given by the Lebesgue measure on each S_l so that $\Lambda(S_{1-1/n}) = 1/n$, and $\Lambda(\{*\} \times S_1) = 1$ for the fiber of B_1 . By Corollary 3.16,

$$\text{Cat}_{\text{meas}}(\mathcal{F}, \Lambda) = 2 \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{3}.$$

This transverse invariant measure is not finite in compact sets and is not regular. Observe that this lamination is not compact-Hausdorff, however it is easy to prove that $\text{Cat}_{\text{top}}(\mathcal{F}, \Lambda) = \infty$.

3. Dimensional upper bound

A topological version of the dimensional upper bound for the tangential category is given in this section. The classical result due to W. Singhof and E. Vogt [33] states that, for any C^2 lamination \mathcal{F} on a compact manifold, $\text{Cat}(\mathcal{F}) \leq \dim \mathcal{F} + 1$.

PROPOSITION 7.19. *Let U be a tangentially categorical set, T a complete topological transversal, H a tangential contraction for U and $\varepsilon > 0$. Suppose that Λ is finite in compact sets. Then there exists two open sets V, W covering U and tangential contractions H^V, H^W of each V and W respectively such that*

$$H^V(V \times \{1\}) + H^W(W \times \{1\}) \leq \Lambda(T) + \varepsilon$$

PROOF. Notice that $T_H = H(U \times \{1\})$ is a transverse set (Lemma 7.3); in fact, for any $x \in U$, there exists an \mathcal{F}_U -saturated open neighborhood $U_x \subset U$ such that $H(U_x \times \{1\})$ is a transversal associated to some foliated chart. Hence, since T is a complete transversal, by using Lemma 7.15, there exists a partition $\{F, V\}$ of U such that F is closed in U and null transverse, and V is an open set so that there exist a tangential contraction $H^V : V \times \mathbb{R} \rightarrow \mathcal{F}$ satisfying $H(V \times \{1\}) \subset T$. Now, observe that $\Lambda(H(F \times \{1\})) = 0$. Since Λ is externally regular on σ -compact sets (by the finiteness on compact sets), there exists an open neighborhood S of $H(F \times \{1\})$ in T_H such that $\Lambda(S) < \varepsilon$. Let $W = H(-, 1)^{-1}(S)$. Clearly, these V and W satisfy the stated conditions. \square

THEOREM 7.20. *Let $(M, \mathcal{F}, \Lambda)$ be a C^2 foliated compact manifold with a transverse invariant measure that is finite in compact sets. Let T be a complete transversal of \mathcal{F} . Then*

$$\text{Cat}(\mathcal{F}, \Lambda) \leq (\dim \mathcal{F} + 1) \cdot \Lambda(T).$$

PROOF. Since $\text{Cat}(\mathcal{F}) \leq \dim \mathcal{F} + 1$, we can use Proposition 7.19 with $\dim \mathcal{F} + 1$ tangentially categorical open sets covering M . Then

$$\text{Cat}(\mathcal{F}, \Lambda) \leq (\dim \mathcal{F} + 1) \cdot (\Lambda(T) + \varepsilon)$$

for all $\varepsilon > 0$. \square

COROLLARY 7.21. *Let (X, \mathcal{F}) a minimal foliated manifold and let Λ be a transverse invariant measure of \mathcal{F} that is regular without atoms. Then $\text{Cat}(\mathcal{F}, \Lambda) = 0$.*

REMARK 23. The same arguments showed here can be used to prove the following more general result. Let $N \geq \text{Cat}(\mathcal{F})$ and let T be a complete transversal, then $\text{Cat}(\mathcal{F}, \Lambda) \leq N \cdot \Lambda(T)$, and this holds without further assumptions in the structure of the lamination (neither compactness nor any differentiable structure is needed).

4. Transverse invariance of the nullity of the Λ -category

In this section, we show that the positivity or nullity of the Λ -category depends only on the holonomy pseudogroup and the transverse invariant measure; in fact, it can be considered as a property of pseudogroups with invariant measures, which is preserved by (measure preserving) equivalences of pseudogroups. We refer to [16, 36] for the basic notions on pseudogroups. We recall here the definition of pseudogroup of homeomorphisms, equivalence of pseudogroups, and a definition of a good set of generators. Then we define the Λ -category of pseudogroups with invariant measures, showing that its nullity or positivity is preserved by (measure preserving) equivalences of pseudogroups, and it gives a lower bound for the Λ -category of foliations when the holonomy pseudogroup is considered.

First of all, we recall a result of the theory of topological dimension due to Milnor (see e.g. [21]), which will be useful in the following sections and chapters of this work.

PROPOSITION 7.22 (Dimensional trick [21]). *Let X be a paracompact space of finite topological dimension and let \mathcal{U} be an open covering of X . Then there exists a covering $\{V_0, \dots, V_{\dim X}\}$ of X such that each V_i is a union of a countable family of disjoint open sets V_{ij} and the open covering $\{V_{ij}\}$ is a refinement of \mathcal{U} .*

DEFINITION 7.23 (Pseudogroup). Let T be a topological space. Let Γ be a set of homeomorphisms between open subsets of T such that $\text{id}_T \in \Gamma$. It is said that Γ is a *pseudogroup* if it is closed by the operations of composition (wherever defined), inversion, restriction to open sets and combination.

Pseudogroups are generalizations of groups of transformations (or actions, or dynamical systems), and the most basic dynamical concepts have obvious versions for pseudogroups: orbits, saturated sets, etc. The restriction of a pseudogroup Γ on T to an open subset $U \subset T$ is the pseudogroup that consists of all elements of Γ whose domain and image is contained in U . We suppose for the rest of the section that T is a paracompact space of finite topological dimension and Γ is finitely generated.

DEFINITION 7.24 (Equivalence of pseudogroups). Let Γ and Γ' be pseudogroups of local homeomorphisms of topological spaces T and T' , respectively. An (*étalé*) *morphism* $\Phi : \Gamma \rightarrow \Gamma'$ is a set of homeomorphisms from open sets of T to open sets of T' such that $\phi \circ g \circ \psi^{-1} \in \Gamma'$ for all $\phi, \psi \in \Phi$ and $g \in \Gamma$. Φ is called an *equivalence* if $\Phi^{-1} = \{\phi^{-1} \mid \phi \in \Phi\}$ is a morphism $\Gamma' \rightarrow \Gamma$; in this case, Γ and Γ' are said to be *equivalent*. The existence of an equivalence between Γ and Γ' is equivalent to the existence of a pseudogroup Γ'' on $T \sqcup T'$ such that T and T' meet all Γ'' -orbits, and the restrictions of Γ'' to T and T' are Γ and Γ' .

DEFINITION 7.25. Let S be a symmetric set of generators of Γ . For each $n \in \mathbb{N}$, let S_n denote the set of compositions of n elements in S , and let $S_\infty = \bigcup_n S_n$. A *deformation* of U is a map $h : U \rightarrow T$ that is combination of maps $h_i : U_i \rightarrow T$ in S_∞ restricted to disjoint open sets; we may use the notation $h \equiv (h_i)$. It is said that U is *deformable* if there is a deformation of U . The pairs (U_i, h_i) are called *components* of h .

Let Λ be a Γ -invariant measure on T .

DEFINITION 7.26. The Λ -category of a pseudogroup with an invariant measure, (T, Γ, Λ) , and a symmetric set of generators S is defined as

$$\text{Cat}(\Gamma, \Lambda, S) = \inf_{\mathcal{U}, h^{\mathcal{U}}} \sum_{U \in \mathcal{U}} \Lambda(h^U(U)) ,$$

where \mathcal{U} runs in the collection of open coverings of T by deformable sets, and $h^{\mathcal{U}} \equiv (h_i^{\mathcal{U}})$ runs in the family of deformations of each $U \in \mathcal{U}$.

PROPOSITION 7.27. $\text{Cat}(\Gamma, \Lambda, S) = 0$ if and only if, for any $\varepsilon > 0$, there exists an open set $U \subset T$ meeting any Γ -orbit and $\Lambda(U) < \varepsilon$.

PROOF. It is clear that $\text{Cat}(\Gamma, \Lambda, S) = 0$ means the existence of such open sets. To prove the reciprocal, let U be an open set meeting any Γ -orbit and such that $\Lambda(U) < \varepsilon$. By Proposition 7.22, there exists a covering $\{U_0, \dots, U_{\dim T}\}$ of T by deformable open sets in U and deformations $h_i : U_i \rightarrow T$ such that $(h_i(U_i)) \subset U$. Therefore $\text{Cat}(\mathcal{F}, \Lambda) \leq (\dim X + 1)\varepsilon$. \square

COROLLARY 7.28. The nullity or positivity of $\text{Cat}(\Gamma, \Lambda, S)$ is independent of the choice of S .

PROOF. By Proposition 7.27, we have a characterization of $\text{Cat}(\mathcal{F}, \Lambda, S)$ by a condition on the measure of open sets that meet all orbits, which does not depend on the choice of generators. \square

According to Corollary 7.28, instead of $\text{Cat}(\mathcal{H}, \Lambda, S)$, the notation $\text{Cat}(\Gamma, \Lambda)$ will be for the conditions $\text{Cat}(\Gamma, \Lambda) = 0$ or $\text{Cat}(\Gamma, \Lambda) > 0$, which makes sense without any reference to the system of generators.

COROLLARY 7.29. If Λ is a regular measure, then the Λ -category of finitely generated pseudogroups on paracompact spaces of finite topological dimension is invariant by measure preserving equivalences.

PROOF. From the regularity of the measures, it easily follows that the existence of an open subset meeting all orbits with arbitrarily small measure is invariant by measure preserving pseudogroup equivalences. So the result follows from Proposition 7.27. \square

Consider a foliated manifold (M, \mathcal{F}) with a transverse invariant measure Λ , and let Γ denote its holonomy pseudogroup acting on a complete transversal T associated to a regular foliated atlas \mathcal{U} of \mathcal{F} . A symmetric set of generators, $E^{\mathcal{U}}$, is given by all holonomy maps induced by any non-empty intersection of a pair of charts in \mathcal{U} .

PROPOSITION 7.30. Let Λ be a measure that is finite on compact sets and Γ -invariant on T . Then $\text{Cat}(\Gamma, \Lambda, E^{\mathcal{U}}) \leq \text{Cat}(\mathcal{F}, \Lambda)$.

PROOF. Let $\{U_n \mid n \in \mathbb{N}\}$ be a covering of M by tangentially categorical open sets and $H^n : U_n \times \mathbb{R} \rightarrow \mathcal{F}$ be tangential deformations such that $\sum_n \Lambda(H^n(U_n \times \{1\})) < \text{Cat}(\mathcal{F}, \Lambda) + \varepsilon$. Without loss of generality, we can suppose that $H^n(U_n \times \{1\}) \subset T$ by using Lemma 7.15. Hence $\{U_n \cap T \mid n \in \mathbb{N}\}$ is a covering of T and $H^n(-, 1) : U_n \cap T \rightarrow T$ is a holonomy deformation. \square

REMARK 24. In Proposition 7.30, the reverse inequality does not hold in general, as shown by the following simple example. Consider a one compact leaf foliation; for instance, S^1 . A complete transversal is given by one point $z \in S^1$, and the corresponding holonomy pseudogroup is $\Gamma = \{\text{id}\}$. Choose the transverse invariant measure Λ given by $\Lambda(\{z\}) = 1$. Then $\text{Cat}(\Gamma, \Lambda) = 1$, whilst $\text{Cat}(S^1, \Lambda) = \text{Cat}(S^1) = 2$.

PROPOSITION 7.31. *Let $(M, \mathcal{F}, \Lambda)$ be a foliated manifold of finite dimension with a transverse invariant measure, and let Γ be its holonomy pseudogroup. If $\text{Cat}(\Gamma, \Lambda) = 0$, then $\text{Cat}(\mathcal{F}, \Lambda) = 0$.*

PROOF. Let $\mathcal{U} = \{U_n\} \in \mathbb{N}$ be a regular foliated atlas, and let T_j be a transversal associated to each foliated chart $U_j \in \mathcal{U}$. For $\delta > 0$, let $\{V^k\}_{k \in \mathbb{N}}$ be an open covering of T and let $h^k \equiv (h_i^k)$ be a deformation of each V^k such that $\sum_k \Lambda(\bigcup_i h_i^k(V_i^k)) < \delta$. Let $\text{sat}_j(B)$ denote the saturation of B in the chart $U_j \in \mathcal{U}$. Let $\dim M = m$.

By Proposition 7.22, there exists a refinement \mathcal{B} of the covering $\{\text{sat}_j(T_j \cap V^k)\}_{j,k}$ such that any point in M is contained in at most $m + 1$ sets in \mathcal{B} , and there exists some point meeting $m + 1$ sets in \mathcal{B} . We can take \mathcal{B} so that it can be subdivided into $m + 1$ families $\mathcal{B}_0, \dots, \mathcal{B}_{\dim M}$ of mutually disjoint open sets. Set $D_i = \bigcup_{B \in \mathcal{B}_i} B$.

Each connected component of D_i is contained in some of the open sets $\text{sat}_j(T_j \cap V^k)$, $1 \leq j \leq K$, $k \in \mathbb{N}$. Now, it is easy to define a tangential contraction H^i for each D_i such that $\Lambda(H^i(D_i \times \{1\})) < \delta$. It is enough to define H^i on each connected component of D_i since each connected component is contained in some $\text{sat}_j(T_j \cap V^k)$. Choose the minimum k satisfying this condition for some j , and, then, choose the minimum j satisfying this condition for that k . By connectivity, every connected component of D_i is contained in only one of the open sets $\text{sat}_j(T_j \cap V_l^k)$ for $l \in \mathbb{N}$. Hence it can be tangentially contracted into the transversal V_l^k , and then it can be deformed to $h_l^k(V_l^k)$ by the tangential homotopy $G_l^k : V_l^k \times [0, 1] \rightarrow M$ associated to the holonomy map h_l^k . With this tangential homotopy, it is clear that $\sum_i \Lambda(H^i(D_i \times \{1\})) < \sum_{k \in \mathbb{N}} \Lambda(\bigcup_i h_i^k(V_i^k)) < \delta$. Therefore $\text{Cat}(\mathcal{F}, \Lambda) < (m + 1)\delta$. \square

REMARK 25. By Corollary 7.29, the statement of Proposition 11.13 is indeed valid for any representative of the holonomy pseudogroup.

COROLLARY 7.32. *The nullity or positivity of the Λ -category of a foliated manifold of finite dimension with a regular transverse invariant measure Λ , which is finite on compact sets, is an invariant of the holonomy pseudogroup and the transverse invariant measure.*

CHAPTER 8

Upper semicontinuity of the topological Λ -category

We adapt a result of W. Singhof and E. Vogt [33], which asserts that the tangential category is an upper semicontinuous map defined on the space of C^2 foliations over a C^∞ closed manifold M .

1. Introduction and notations

Consider an embedding of M into an Euclidean space \mathbb{R}^N . Then each C^2 foliation on M can be identified to a C^1 -map $M \rightarrow \mathbb{R}^{N^2}$, which maps each point to the orthogonal projection of \mathbb{R}^N to the tangent space of the foliation at this point. The topology on the space of C^{k+1} foliations over M is induced by the topology of the C^k -maps $M \rightarrow \mathbb{R}^{N^2}$. This topological space is denoted by $\text{Fol}_p^k(M)$.

Let $T \pitchfork \mathcal{F}$ denote that T is transverse to \mathcal{F} (in the differentiable sense).

PROPOSITION 8.1 (Singhof-Vogt [33]). *Let M be a closed n -submanifold of \mathbb{R}^N and $\mathcal{F} \in \text{Fol}_p^1(M)$. Let A_1, \dots, A_m be compact C^1 $(n-p)$ -submanifolds of M transverse to \mathcal{F} , let $C_i \subset A_i$ be a collar of the boundary of each A_i , and let $A = \bigcup_i (A_i \setminus C_i)$. Then there exists a neighborhood \mathcal{V} of \mathcal{F} in $\text{Fol}_p^1(M)$ and an open neighborhood W of A in M such that W is \mathcal{G} -categorical for all $\mathcal{G} \in \mathcal{V}$.*

PROPOSITION 8.2 (Singhof-Vogt [33]). *Let $\mathcal{F} \in \text{Fol}_p^1(M)$, let U be an open set in M , and let $F : U \times I \rightarrow M$ be a tangential homotopy such that $F(x, 0) = x$ for all $x \in U$. Let $\varepsilon > 0$ and let K be a compact subset of U . Then there exists a neighborhood \mathcal{V} of \mathcal{F} such that, for all $\mathcal{G} \in \mathcal{V}$, there exists a \mathcal{G} -tangential homotopy $G : U \times I \rightarrow M$ with $G(x, 0) = x$ for all $x \in U$ and $|F - G| < \varepsilon$ in $K \times I$.*

In the next proposition, B_r^k denotes the open ball in \mathbb{R}^k of radius r and centered at the origin, and D_r^k denotes the corresponding closed ball.

PROPOSITION 8.3 (Singhof-Vogt [33]). *Let \mathcal{F} be a C^{r+1} foliation on a closed manifold $M \subset \mathbb{R}^N$ with $\dim \mathcal{F} = p$ and $\dim M = n$, and let $a \in M$. Let $\varphi : B_3^p \times B_4^{n-p} \rightarrow U \subset M$ be a parametrization associated to a foliated chart containing a . Then there exists a neighborhood \mathcal{V} of \mathcal{F} in $\text{Fol}_p^r(M)$ such that, for all $\mathcal{G} \in \mathcal{V}$ and $t \in B_3^{n-p}$, there exists a map $g_t : B_3^p \rightarrow B_4^{n-p}$ with the following properties:*

- (i) *The map $g : B_3^p \times B_3^{n-p} \rightarrow B_3^p \times B_4^{n-p}$, defined by $(x, t) \mapsto (x, g_t(x))$, is a C^{r+1} -embedding.*
- (ii) *$B_3^p \times B_2^{n-p}$ is contained in the image of g and $B_3^p \times B_1^{n-p}$ is contained in $g(B_3^p \times B_2^{n-p})$.*

- (iii) For each $t \in B_3^{n-p}$, the set $\{(x, g_t(x)) \mid x \in B_3^p\}$ is contained in the leaf of the pull back $\varphi^*\mathcal{G}$ of \mathcal{G} to $B_3^p \times B_4^{n-p}$ through the point $(0, t)$.

In fact, g is uniquely determined by (i) and (iii), the map defined by $\varphi \circ g : B_3^p \times B_3^{n-p} \rightarrow U_g \subset M$ is a C^{r+1} parametrization of a foliated chart for \mathcal{G} , and $\varphi(B_3^p \times B_2^{n-p}) \subset U_g$, where $U_g = \varphi \circ g(B_3^p \times B_3^{n-p})$.

DEFINITION 8.4 (Singhof-Vogt [33]). Under the conditions of Proposition 8.3, the family $\{\varphi \circ g \mid \mathcal{G} \in \mathcal{V}\}$ is called a *simultaneous local parametrization* of \mathcal{V} in M .

Simultaneous local parametrizations allow to work with foliations near other ones in an easy way. We will use them to prove an important lemma of this section.

EXAMPLE 8.5. A previous motivation of the upper semicontinuity of the topological Λ -category is given by the Krönecker flows \mathcal{F}_α on the torus, where $\alpha \in \mathbb{R}$ is the slope of the flow lines. The topological Λ -category of \mathcal{F}_α can be easily computed by using Proposition 7.18 if α is rational, and it is zero if α is irrational by Corollary 7.21. We restrict our study to the subspace of these flows, $\{\mathcal{F}_\alpha \mid \alpha \in \mathbb{R}\} \subset \text{Fol}_1^\infty(T^2)$, which is homeomorphic to \mathbb{R} by the mapping $\mathcal{F}_\alpha \mapsto \alpha$. A transverse invariant measure Λ for all of these flows is the normalized Lebesgue measure on a fixed meridian. The topological Λ -category is given by the mapping $\frac{m}{n} \mapsto \frac{2}{|n|}$, $0 \mapsto 2$ and $r \mapsto 0$, where m and n are coprime integers and r is irrational. Clearly, this map is upper semicontinuous.

In the same example, we can also allow to vary the measure in a continuous way by taking $(\mathcal{F}_\alpha, f(\alpha) \cdot \Lambda)$, where Λ is the normalized Lebesgue measure and $f : \mathbb{R} \rightarrow \mathbb{R}^+$ a continuous map. The Λ -category function, in this case, is the product of the above map and f , which is upper semicontinuous as well.

Example 8.5 suggests that the Λ -category is an upper semicontinuous function in a certain topological space of foliations with transverse invariant measures. From now on in this section, suppose that the transverse invariant measures are finite on compact sets.

REMARK 26. If S is a complete differentiable transversal of a foliation \mathcal{F} , then there exists a neighborhood $U^S(\mathcal{F})$ of \mathcal{F} in $\text{Fol}_p^1(M)$ such that S is a complete differentiable transversal of all $\mathcal{G} \in U^S(\mathcal{F})$.

2. Strong and weak topology

Now, we define two topologies in the space of foliations with transverse invariant measure. They are suitable modifications of the space $\text{Fol}_p^1(M)$.

DEFINITION 8.6 (Strong topology). Let (\mathcal{F}, Λ) be a C^2 foliation of dimension p on a closed manifold M with a transverse invariant measure. Let \mathcal{U} be a neighborhood of \mathcal{F} in $\text{Fol}_p^1(M)$, T a differentiable complete transversal of \mathcal{F} , and $\varepsilon > 0$. Let

$$U(\mathcal{F}, \Lambda, \mathcal{U}, T, \varepsilon) = \{(\mathcal{G}, \Delta) \mid \mathcal{G} \in \mathcal{U}, T \pitchfork \mathcal{G}, \|\Lambda_T - \Delta_T\| < \varepsilon\},$$

where (\mathcal{G}, Δ) is a C^2 foliation of dimension p on M with a transverse invariant measure. This kind of sets form a base for a topology in the space of C^2 foliations of dimension p on M with transverse invariant measures finite on compact sets. This topological space is denoted by $\text{MeasFol}_p^1(M)$.

For locally compact Polish spaces, there exists a weak topology on the set of measures coarser than the norm topology. We use it to give a weaker topology in the above space. It is said that a sequence $\{\Lambda_n\}$ of measures on a locally compact Polish space P converges in the weak sense to a measure Λ when the sequence $\int_P f d\Lambda_n$ converges to $\int_P f d\Lambda$ for all continuous function $f : P \rightarrow \mathbb{R}$.

DEFINITION 8.7 (Weak topology). Let $(\mathcal{F}_n, \Lambda_n)$ be a sequence of C^2 foliations with transverse invariant measures on a closed manifold M . We say that $(\mathcal{F}_n, \Lambda_n)$ converges in the weak sense to (\mathcal{F}, Λ) if \mathcal{F}_n converges to \mathcal{F} in $\text{Fol}_p^1(M)$ and, for all complete differentiable transversal T to \mathcal{F} and for all continuous function $f : T \rightarrow \mathbb{R}$, the sequence $\int_T f d\Lambda_n$ converges to $\int_T f d\Lambda$. The topological space determined by this condition will be noted by $\mathcal{W}\text{MeasFol}_p^1(M)$.

REMARK 27. Since measures are finite on compact sets, they are externally regular on σ -compact sets. It is also true that they are *internally regular* on open sets; i.e., for all open set V contained in a transversal of \mathcal{F} , we have

$$\Lambda(V) = \max\{ \Lambda(K) \mid K \subset V \text{ is a compact subset} \}.$$

We prove upper semicontinuity of the topological Λ -category with respect to the weak topology, which implies its upper semicontinuity with respect to the strong topology too.

3. Upper semicontinuity of the Λ category

DEFINITION 8.8. Let $f : A \rightarrow B$ be a map. For $n \in \mathbb{N} \cup \{\infty\}$, the n -preimage of f is

$$F_n^f = \{ x \in A \mid \#f^{-1}(f(x)) = n \}.$$

In Definition 8.8, the sets F_n^f are σ -compact when f is a continuous map, and therefore they are measurable.

DEFINITION 8.9. A *local differentiable embedded transversal* is a transversal associated to a differentiable foliated chart; equivalently, it is a differentiable embedding $i : B^{n-p}(r) \rightarrow M$ such that $i(B^{n-p}(r))$ is contained in a distinguished open set and it is diffeomorphic to an open set of an associated transversal via the projection map. An open set U in a foliated space is called *regular* when there exist a finite number of local differentiable embedded transversals, T_1, \dots, T_k , contained in U such that $T_1 \cup \dots \cup T_k$ is a complete transversal of \mathcal{F}_U . Suppose \mathbb{R}^2 with the usual foliation by lines $\mathbb{R} \times \{*\}$, consider the closed set $F = (\{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}) \times [0, \infty)$, the open set $\mathbb{R}^2 \setminus F$ is not regular.

REMARK 28. Let U an open set in M and $K \subset U$ a compact set. Then there exists a regular open set V such that $K \subset V$ and $\overline{V} \subset U$. To prove this assertion, it is enough to cover K with a finite number of foliated charts whose closures are contained in U .

REMARK 29. Let (M, \mathcal{F}) be a C^∞ foliated closed manifold, let U be a regular \mathcal{F} -categorical open set and let H be an \mathcal{F} -contraction for U . By using Lemma 7.3, there exist transversals T_1, \dots, T_k satisfying the following conditions:

- (1) Any leaf of \mathcal{F}_U meets each T_i at most in one point and cuts some of them.
- (2) $H(T_i \times \{1\})$ is a local embedded transversal of \mathcal{F} .

By general position arguments, it is easy to prove that there exists an \mathcal{F} -contraction G such that $\Lambda(G(U \times \{1\})) \leq \Lambda(H(U \times \{1\}))$ and $G(T_i \times \{1\})$ is a local differentiable embedded transversal of \mathcal{F} for $1 \leq i \leq k$. Therefore we also suppose that

- (3) $H(T_i \times \{1\})$ is a differentiable transversal of \mathcal{F} .

Hence $H(U \times \{1\}) = \bigcup_{i=1}^k H(T_i \times \{1\})$.

The following lemma is obvious since $H(-, 1) : \bigcup_{i=1}^k T_i \rightarrow M$ is a continuous map.

LEMMA 8.10. *Let U be a regular \mathcal{F} -categorical open set, let H be an \mathcal{F} -contraction for U , let T_1, \dots, T_k be transversals satisfying the previous conditions, and let Λ be a transverse invariant measure. Then*

$$\tilde{\Lambda}(H(U \times \{1\})) = \sum_{i=1}^k \tilde{\Lambda}(H(F_i^{H(-,1)} \times \{1\})) = \sum_{i=1}^k \frac{1}{i} \Lambda(F_i^{H(-,1)}),$$

where we consider $H(-, 1) : \bigcup_{i=1}^k T_i \rightarrow M$, and $F_i^{H(-,1)}$ denotes the i -preimage of $H(-, 1)$.

LEMMA 8.11. *Let U be a regular \mathcal{F} -categorical open set, let $H : U \times I \rightarrow M$ be an \mathcal{F} -contraction, let T_1, \dots, T_k be transversals satisfying conditions of Remark 29, and let Λ be a transverse invariant measure. Let O be an open set containing $H(U \times \{1\}) = \bigcup_{i=1}^k H(T_i \times \{1\})$ such that each leaf of \mathcal{F}_O meets $H(U \times \{1\})$. Let G be an \mathcal{F} -contraction for O . Then*

$$\tilde{\Lambda}(H(U \times \{1\})) \geq \tilde{\Lambda}(H * G(U \times \{1\})).$$

PROOF. Observe that $\bigcup_{i=1}^j F_i^{H(-,1)} \supset \bigcup_{i=1}^j F_i^{H * G(-,1)}$ for $1 \leq j \leq k$, and then use Lemma 8.10. \square

The following lemma has an analogous proof.

LEMMA 8.12. *Under the same hypothesis, let K be a σ -compact subset of O . Then*

$$\tilde{\Lambda}(H(U \times \{1\})) \geq \tilde{\Lambda}(G(K \times \{1\})).$$

PROPOSITION 8.13. *Let $(\mathcal{F}, \Lambda) \in \mathcal{W}\text{MeasFol}_p^1(M)$, let U be a regular \mathcal{F} -categorical open set, let $H : U \times I \rightarrow M$ be an \mathcal{F} -contraction and let W be an open set such that $\overline{W} \subset U$. Let T_1, \dots, T_k be transversals satisfying*

the conditions of Remark 29, let K be a closed or open set in $\bigsqcup_{i=1}^k T_i$, and let $\varepsilon, \delta > 0$. Then, for every sequence $(\mathcal{F}_n, \Lambda_n)$ converging to (\mathcal{F}, Λ) in $\mathcal{W}\text{MeasFol}_p^1(M)$, there exists $N \in \mathbb{N}$ such that, $\forall n \geq N$,

- (i) the transversals $T_1, \dots, T_k, H(T_1 \times \{1\}), \dots, H(T_k \times \{1\})$ of \mathcal{F} are also transversals of \mathcal{F}_n ;
- (ii) $|\Lambda(K) - \Lambda_n(K)| < \varepsilon$; and
- (iii) there exists a C^1 tangential homotopy $H^{\mathcal{F}_n}$ for \mathcal{F}_n defined on \overline{W} for all n such that $|H(x, t) - H^{\mathcal{F}_n}(x, t)| < \delta$ for $x \in \overline{W}$.

PROOF. Clearly, (i) is a consequence of Remark 26. Observe that (ii) is obtained by using Urysohn's lemma and the regularity of the measure (see Remark 27). Finally, (iii) is a direct consequence of Proposition 8.2. \square

The following assertion is easy to prove. In the following, we use the notation $A_i = H(T_i \times \{1\})$.

LEMMA 8.14. *There exists a collar C_i of the boundary of A_i such that $H(\overline{W} \times \{1\}) \subset \bigcup_{i=1}^k A_i \setminus C_i$.*

Let P be an open set containing $\bigcup_{i=1}^k A_i \setminus C_i$ under the conditions of Proposition 8.1. By taking N larger if necessary, P is \mathcal{F}_n -categorical for all $n \geq N$. Let $G^{\mathcal{F}_n}$ be an \mathcal{F}_n -contraction for P .

PROPOSITION 8.15. *For N large enough, there exists an open set $P' \subset P$ such that $H(\overline{W} \times \{1\}) \subset P'$ and each leaf of $\mathcal{F}_n|_{P'}$ meets $\bigcup_{i=1}^k A_i \setminus C_i$ for all $n \geq N$.*

The proof of Proposition 8.15 is technical and will be given at the end of this section.

LEMMA 8.16. *Let U be a regular \mathcal{F} -categorical open set, let H be an \mathcal{F} -contraction of U and let W be an open set such that $\overline{W} \subset U$. Then, for all $\varepsilon > 0$ and for all sequence $(\mathcal{F}_n, \Lambda_n)$ converging to (\mathcal{F}, Λ) in $\mathcal{W}\text{MeasFol}_p^1(M)$, there exists some $N \in \mathbb{N}$ and there exists an \mathcal{F}_n -contraction J^n of W for all $n \geq N$ such that*

$$\tilde{\Lambda}(J^n(W \times \{1\})) \leq \tilde{\Lambda}(H(U \times \{1\})) + \varepsilon.$$

PROOF. By Remark 29, Lemma 8.14 and Propositions 8.15 and 8.13, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, there is a tangential homotopy H^n for \mathcal{F}_n with $H^n(W \times \{1\}) \subset P'$, where P' is the open set given by Proposition 8.15. Moreover there exists an \mathcal{F}_n -contraction G^n of P' for $n \geq N$. We also can suppose

$$\left| \Lambda_n \left(\bigcup_{i=1}^k A_i \right) - \Lambda \left(\bigcup_{i=1}^k A_i \right) \right| < \varepsilon$$

for $n \geq N$ by Remark 29-(3). Therefore, by Lemma 8.12,

$$\begin{aligned} \Lambda(H(U \times \{1\})) + \varepsilon &= \Lambda\left(\bigcup_{i=1}^k H(T_i \times \{1\})\right) + \varepsilon \\ &\geq \Lambda_n\left(\bigcup_{i=1}^k H(T_i \times \{1\})\right) \geq \Lambda_n\left(\bigcup_{i=1}^k (A_i \setminus C_i)\right) \\ &\geq \Lambda_n\left(\bigcup_{i=1}^k P' \cap (A_i \setminus C_i)\right) \geq \Lambda_n\left(\bigcup_{i=1}^k H^n * G_1^n(W)\right). \quad \square \end{aligned}$$

THEOREM 8.17 (Upper semicontinuity of the topological Λ -category). *The topological Λ -category map, $\text{Cat} : \mathcal{W}\text{MeasFol}_p^1(M) \rightarrow \mathbb{R}$, is upper semicontinuous.*

PROOF. We have to prove that, for all $\varepsilon > 0$ and for any sequence $(\mathcal{F}_n, \Lambda_n)$ converging to (\mathcal{F}, Λ) in $\mathcal{W}\text{MeasFol}_p^1(M)$, there exists $N \in \mathbb{N}$ such that $\text{Cat}(\mathcal{F}_n, \Lambda_n) \leq \text{Cat}(\mathcal{F}, \Lambda) + \varepsilon$ for all $n \geq N$. Take a finite covering $\{U_1, \dots, U_K\}$ of M by \mathcal{F} -categorical open sets of M , and let H^1, \dots, H^K be \mathcal{F} -contractions such that

$$\sum_{i=1}^K \Lambda(H^i(U_i \times \{1\})) \leq \text{Cat}(\mathcal{F}, \Lambda) + \frac{\varepsilon}{2}.$$

By paracompactness and Remark 28, we can suppose that each U_i is regular. By paracompactness again, there is an open covering $\{W_i\}_{i=1, \dots, K}$ of M such that $\overline{W_i} \subset U_i$. By Lemma 8.16, there exists $N \in \mathbb{N}$, and there are \mathcal{F}_n -contractions $H^{n,i} : W_i \times I \rightarrow M$ for $1 \leq i \leq K$ and $n \geq N$ such that

$$\Lambda_n(H^{n,i}(W_i \times \{1\})) \leq \Lambda(H^i(U_i \times \{1\})) + \frac{\varepsilon}{2K}.$$

Finally, for all $n \geq N$, we get

$$\begin{aligned} \text{Cat}(\mathcal{F}_n, \Lambda_n) &\leq \sum_{i=1}^K \Lambda_n(H^{n,i}(W_i \times \{1\})) \\ &\leq \sum_{i=1}^K \Lambda(H^i(U_i \times \{1\})) + \frac{\varepsilon}{2} \leq \text{Cat}(\mathcal{F}, \Lambda) + \varepsilon. \quad \square \end{aligned}$$

Now, we prove Proposition 8.15 to conclude this section. We restrict to the case where T is a local differentiable embedded transversal contained in a simultaneous local parametrization. This case easily implies the general case.

Let us introduce a metric on $\text{Fol}_p^1(M)$. Remember that a C^2 foliation \mathcal{F} can be identified to a C^1 -map $\mathcal{F} : M \rightarrow \mathbb{R}^{N^2} \equiv \mathcal{L}(\mathbb{R}^N, \mathbb{R}^N)$. For each $x \in M$, the corresponding linear map $\mathcal{F}_x : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is the orthogonal projection of \mathbb{R}^N onto $T_x \mathcal{F}$, where an embedding of M into \mathbb{R}^N is considered, and $T_x \mathcal{F} \subset T_x \mathbb{R}^N \equiv \mathbb{R}^N$ denotes the subspace tangent to \mathcal{F} at x .

DEFINITION 8.18. For foliations \mathcal{F} and \mathcal{G} on M with the same dimension, let

$$d(\mathcal{F}, \mathcal{G}) = \max_{x \in M} \|\mathcal{F}_x - \mathcal{G}_x\|.$$

This map d is called the *foliated metric* on $\text{Fol}_p^1(M)$.

REMARK 30. Clearly, d is a metric on $\text{Fol}_p^1(M)$ that induces its topology; i.e., $\mathcal{F}_n \rightarrow \mathcal{F}$ in $\text{Fol}_p^1(M)$ if and only if $d(\mathcal{F}_n, \mathcal{F}) \rightarrow 0$.

Since Proposition 8.15 is of local nature, we interpret the foliated metric from the point of view of simultaneous local parametrizations. Consider the notation of Proposition 8.3. Let $\Phi = \{\varphi \circ g \mid \mathcal{G} \in \mathcal{V}\}$ be a simultaneous local parametrization around a point $a = \varphi(0, 0) \in M$, according to Definition 8.4.

DEFINITION 8.19. Let $\mathcal{F}, \mathcal{G} \in \mathcal{V}$. Define the φ -*foliated metric* on $\varphi \in \Phi$ by

$$d_\varphi(\mathcal{F}, \mathcal{G}) = \max_{x \in D_3^p \times D_1^{n-p}} \|\mathcal{F}_x - \mathcal{G}_x\|.$$

Obviously, \mathcal{F}_n converges to \mathcal{F} in $\text{Fol}_p^1(M)$ if and only if it converges to \mathcal{F} with respect to the foliated metric on any simultaneous local parametrization relative to \mathcal{F} . Therefore we can restrict to the simple case where $M = \mathbb{R}^n$ and \mathcal{F} is the standard foliation by parallel planes with dimension p . For this, we take a foliated diffeomorphism $\alpha : \text{int}(D_3^p \times D_1^{n-p}) \rightarrow \mathbb{R}^n$ with $\alpha(0) = 0$.

DEFINITION 8.20. Let \mathcal{F} be the foliation of dimension p on \mathbb{R}^n defined by planes parallel to the plane $x_{p+1} = \dots = x_n = 0$. For all $\varepsilon, \delta > 0$, let

$$\begin{aligned} R_0(\varepsilon, \delta) \\ = \{ (x_1, \dots, x_n) \mid \|(x_{p+1}, \dots, x_n)\| \leq \varepsilon \|(x_1, \dots, x_p)\|, \|(x_1, \dots, x_n)\| \leq \delta \}. \end{aligned}$$

For $a \in M$, let $R_a(\varepsilon, \delta) = a + R_0(\varepsilon, \delta)$.

REMARK 31. We have $d(\mathcal{F}, \mathcal{G}) < \varepsilon$ if and only if $T_a \mathcal{G} \subset R_a(\varepsilon, \infty) \forall a \in \mathbb{R}^n$.

The following Lemma is easy to prove.

LEMMA 8.21. Let $f : \mathbb{R}^p \rightarrow \mathbb{R}^{n-p}$ be a C^1 map such that $f(0) = 0$ and the graph of its differential map, $df(x)$, is contained in $R_0(\varepsilon, \infty)$ for all $x \in \mathbb{R}^p$. Then the graph of f is contained in $R_0(\varepsilon, \infty)$.

REMARK 32. In our setting, the graph of f represents a leaf near \mathcal{F} . Therefore, for $(v, w) \in B_3^p \times B_1^{n-p}$, there exists a compact region of the form $\overline{\alpha^{-1}(R_{\alpha(v, w)}(\varepsilon, \infty))}$ such that any leaf in $D_3^p \times D_1^{n-p}$ of $\varphi^* \mathcal{G}$ through (v, w) is contained in this region (by taking \mathcal{V} smaller if necessary).

Now, we can prove Proposition 8.15. Consider the particular case $T = \{0\} \times B_{1/2}^{n-p}$, which is an embedded smooth local transversal contained in $B_3^p \times B_1^{n-p}$. Define the ρ -tube of T by $U(T, \rho) = B_\rho^p \times B_{1/2}^{n-p}$, $0 < \rho < 3$. The boundary ∂T is compact, and, for all $x \in \partial T$, let R_x be a region satisfying the conditions of Remark 32. The set $K = \bigcup_{x \in \partial T} R_x$ is compact and therefore closed. The set $P(T, \rho) = U(T, \rho) \setminus K$ is an open neighborhood of T and satisfies the required conditions for a ball centered at \mathcal{F} with small radius with respect to the foliated metric.

PROOF OF PROPOSITION 8.15. Under the conditions of Proposition 8.15, there exist k transversals, S_1, \dots, S_k , such that $\overline{S_i} \subset A_i \setminus C_i$ for $1 \leq i \leq k$, and $H(\overline{W} \times \{1\}) \subset \bigcup_{i=1}^k S_i$. Observe that any smooth embedded local transversal is associated to a foliated chart. Therefore, if P is a neighborhood of $\bigcup_{i=1}^k A_i \setminus C_i$, there exists a neighborhood \mathcal{V} of \mathcal{F} in $\text{Fol}_p^1(M)$ and there exists a neighborhood $P(S_i, \rho_i) \subset P$ of each S_i such that each leaf of $\mathcal{G}_{P(S_i, \rho_i)}$ meets S_i (and therefore meets $A_i \setminus C_i$) for all $\mathcal{G} \in \mathcal{V}$. The open set $\bigcup_{i=1}^k P(S_i, \rho_i)$ satisfies the required conditions. \square

The upper semicontinuity can be a tool to get lower bounds for the Λ -category of a C^2 foliation on a closed C^∞ manifold: if $(\mathcal{F}_n, \Lambda_n) \rightarrow (\mathcal{F}, \Lambda)$ with respect to the weak topology, then $\limsup \text{Cat}(\mathcal{F}_n, \Lambda_n) \leq \text{Cat}(\mathcal{F}, \Lambda)$.

REMARK 33. Measures satisfying the conditions of the Riesz Representation Theorem are regular. Therefore regularity is not a very restrictive condition on the measures.

CHAPTER 9

Critical points

In this chapter, we adapt Theorems 6.8 and 6.9 to the topological case. Like in Section 6, we work with Hilbert laminations. Thus, in the topological setting, the foliated charts are homeomorphisms to $H \times P$, where H is a separable Hilbert space, P is a Polish space (our transverse model). The ambient space of the lamination is separable, and therefore we can work with countable foliated atlases.

We shall work with Hilbert laminations. Thus, the foliated charts are homeomorphisms to $H \times P$, where H is a separable Hilbert space and P is a Polish space. The ambient space of the foliation is a Polish space, and therefore we can work with countable foliated atlases. We hope that the work of this section will be useful to study laminated versions of variational problems where the classical Lusternik-Schnirelmann category was applied [3, 28, 31].

Moreover we consider C^2 Hilbert laminations; i.e., the change of foliated coordinates are leafwise C^2 whose tangential derivatives of order ≤ 2 are continuous on the ambient space. Also, any lamination is assumed to have a locally finite atlas such that each plaque of every chart meets at most one plaque of any other chart. We consider a leafwise Riemannian metric so that its tangential derivatives of order ≤ 2 are continuous on the ambient space; thus each leaf becomes a Riemannian Hilbert manifold. Of course, the holonomy pseudogroup makes sense in this set-up, as well as transverse invariant measures. We only consider regular measures. Here, an open transversal is an embedded space that is locally homeomorphic to a transversal associated to a foliated chart via a projection map.

We consider functions that are C^2 on the leaves whose tangential derivatives of order ≤ 2 are continuous on the ambient space. The functions satisfying the above property are called C^r , and they form a linear space denoted by $C^2(\mathcal{F})$. For a function $f \in C^2(\mathcal{F})$, we set $\text{Crit}_{\mathcal{F}}(f) = \bigcup_{L \in \mathcal{F}} \text{Crit}(f|_L)$. The definition of C^r Hilbert laminations and their C^r functions.

Since any separable Hilbert space admits C^∞ partitions of unity, the transverse model is paracompact and Hausdorff and we consider a locally finite atlas, we have the following.

PROPOSITION 9.1. *Every open cover of a Hilbert lamination of class C^k admits a subordinate partition of unity of class C^k .*

As a consequence of the Proposition 9.1 and following the same argument of [6, 1], we have

PROPOSITION 9.2. *If (X, \mathcal{F}) is a lamination of class C^k , then there is a smooth C^k embedding φ of X in the separable real Hilbert space \mathbb{E} . Moreover,*

a given metric tensor along the leaves can be extended to a metric tensor on \mathbb{E} .

Let (X, \mathcal{F}) be a usual lamination (of finite dimension), of class C^k with $k \geq 1$, embedded in the Hilbert space \mathbb{E} . The restriction of the embedding to each leaf is not an embedding, but only an injective immersion. The smoothness of \mathcal{F} being at least C^1 implies that the map which assigns to a point $x \in X$ the subspace $T_x \mathcal{F}$ of \mathbb{E} is continuous (as a map of \mathcal{F} into the space of n -dimensional subspaces of \mathbb{E}). It follows that if F is a subspace complementary to one $T_x X$ in \mathbb{E} , then it is also complementary to $T_y X$ for y close to x . The key point is that each tangent space $T_x X$ is a finite dimensional subspace of \mathbb{E} . Hence it is closed and has an orthogonal complement.

THEOREM 9.3 ([1]). *Let X be a lamination of finite dimension embedded in \mathbb{E} as above, of class C^2 . Let $i : L \rightarrow \mathbb{E}$ denote the inclusion of a leaf L in $X \subset \mathbb{E}$. Then there is a vector bundle $\pi : N \rightarrow L$ and a neighborhood W of the zero section of N such that the following properties hold:*

- (i) *The map $i : L \rightarrow \mathbb{E}$ extends to a local diffeomorphism $\varphi : W \rightarrow \mathbb{E}$;*
- (ii) *there is a laminated subspace $Y \subset W$, of the same dimension as X , having L as a leaf and transverse to the fibers of N ; and*
- (iii) *as foliated spaces, $Y = \varphi^{-1}(X \cap \varphi(W))$, and the restriction of π to each leaf of Y is a local diffeomorphism into L .*

DEFINITION 9.4 (Tangential isotopy). Let M be a C^r Hilbert manifold, $r \geq 1$. A C^q isotopy on M , $1 \leq q \leq r$, is a C^q differentiable map $\phi : M \times \mathbb{R} \rightarrow M$ such that $\phi_t = \phi(-, t) : M \rightarrow M$ is a diffeomorphism for all $t \in [0, 1]$ and $\phi_0 = \text{id}_M$. If (X, \mathcal{F}) is a C^r Hilbert lamination, a C^q tangential isotopy on (X, \mathcal{F}) is a leafwise C^q map $\phi : X \times \mathbb{R} \rightarrow X$ such that the functions $\phi_t = \phi(-, t) : X \rightarrow X$ are homeomorphisms for all t , and ϕ restricts to usual isotopies on the leaves of \mathcal{F} .

REMARK 34. Let ϕ be a tangential isotopy on X and let $U \subset X$ be an open set. Then $\text{Cat}(U, \mathcal{F}, \Lambda) = \text{Cat}(\phi_t(U), \mathcal{F}, \Lambda)$ and $\text{Cat}(U, \mathcal{F}) = \text{Cat}(\phi_t(U), \mathcal{F})$ for all $t \in \mathbb{R}$.

EXAMPLE 9.5 (Construction of a tangential isotopy [27]). A tangential isotopy can be constructed on a Hilbert manifold by using a C^1 tangent vector field V . There exists a flow $\phi_t(p)$ such that $\phi_0(p) = p$, $\phi_{t+s}(p) = \phi_t(\phi_s(p))$ and $d\phi_t(p)/dt = V(\phi_t(p))$. From the way of obtaining ϕ [27, 10], it follows that the same kind of construction for a C^1 tangent vector field on a measurable Hilbert lamination (X, \mathcal{F}) induces a tangential isotopy on (X, \mathcal{F}) .

Now we obtain a tangential isotopy from the leafwise gradient flow of a differentiable map. It will be modified by a control function α in order to have some control on the deformations induced by the corresponding isotopy. Let ∇f be the gradient tangent vector field of f ; i.e., the unique tangent vector field satisfying $df(v) = \langle v, \nabla f \rangle$ for all $v \in T\mathcal{F}$. Take the C^1 vector field $V = -\alpha(|\nabla f|) \nabla f$, where α is defined just like in Example 6.2. The flow $\phi_t(p)$ of V is defined for $-\infty < t < \infty$ [31], and it is called the *modified gradient flow*.

The partial order relation “ \ll ” for the critical points of f is defined like in the measurable setting, observe that this definition only use the leafwise character of the modified gradient flow.

Let $\gamma(p)$ respectively denote the flow orbit of each point p ; that is, the trace of the curve $\phi_t(p)$ with $t \in \mathbb{R}$.

LEMMA 9.6. *Let $T \subset X$ be a transversal meeting each leaf in a discrete set, and let $\varepsilon > 0$. Then there exists a \mathcal{F} -categorical open set U containing T such that $\text{Cat}(U, \mathcal{F}, \Lambda) < \Lambda(T) + \varepsilon$.*

PROOF. It is easy to see that we can suppose that \bar{T} is contained in a foliated chart V . Let T_V be a transversal associated to V and $\pi : V \rightarrow T_V$ the canonical projection. Since Λ is regular, there exists an open subset $F \subset T_V$ containing $\pi(T)$ such that $\Lambda(F) < \Lambda(\pi(T)) + \varepsilon$. Of course, $T \subset \text{sat}_V(F)$ and $\text{sat}_V(F)$ is an open set that contracts tangentially to F . The result follows since $\Lambda(\pi(T)) \leq \Lambda(T)$. \square

We restrict our study to the case where the leafwise critical points are isolated on the leaves or, in the case of the Λ -category, the set of leaves with non leafwise isolated critical points is null transverse. Therefore the set of leafwise critical points is a transverse set.

LEMMA 9.7. *Suppose that $\text{Crit}_{\mathcal{F}}(f)$ is a transverse set. The modified gradient flow ϕ (see Example 9.5) satisfies the following properties:*

- (i) *The flow runs towards lower level sets of f , i.e., $f(p) \geq f(\phi_t(p))$ for $t > 0$.*
- (ii) *The invariant points of the flow are just the critical points of f .*
- (iii) *A point is critical if and only if $f(\phi_t(p)) = f(p)$ for some $t \neq 0$.*
- (iv) *The points in the α - and ω -limits are critical points if they are non empty.*

PROOF. These properties can be proved in each leaf, considered as a C^2 Hilbert manifold, where (i), (ii) and (iii) follow from the work of J. Schwartz [31].

Under these conditions, the α - and ω -limits are connected sets that consist of critical points if they are non-empty (by using (i), (ii) and (iii)). If $\omega(p)$ is infinite, then all of its points are non-isolated, contradicting the assumption. \square

REMARK 35 (Critical sets). Critical sets are defined like for measurable laminations. Observe that the critical sets are σ -compact in the topological setting.

DEFINITION 9.8. An ω -Palais-Smale (or simply, ω -PS) function is a function $f \in C^2(\mathcal{F})$ such that all of its critical sets are closed (in the ambient topology). all ϕ -orbit has non empty ω -limit and, for any $p \in \text{Crit}_{\mathcal{F}}(f)$, the set $\{x \in \text{Crit}_{\mathcal{F}}(f) \mid p \ll x\}$ is compact, and this set is empty if and only if p is a relative minimum. An α -Palais-Smale (or simply, α -PS) function is defined analogously by taking the set $\{x \in \text{Crit}_{\mathcal{F}}(f) \mid x \ll p\}$.

Of course, f is ω -PS if and only if $-f$ is α -PS. The set of relative minima of a ω -PS function bounded from below is always non-empty in any leaf.

THEOREM 9.9. *Let (X, \mathcal{F}) be a Hilbert lamination endowed with a Riemannian metric on the leaves varying continuously on the ambient space, and let f be an ω -PS function bounded from below. Suppose that critical sets are closed and $\text{Crit}_{\mathcal{F}}(f)$ meets each leaf in a discrete set. Then $\text{Cat}(\mathcal{F}) \leq \#\{\text{critical sets of } f\}$.*

THEOREM 9.10. *Let $(X, \mathcal{F}, \Lambda)$ be a Hilbert lamination endowed with a Riemannian metric on the leaves varying continuously on the ambient space, and with a transverse invariant measure. Let f be an ω -PS function satisfying the hypothesis of Theorem 9.9. Then $\text{Cat}(\mathcal{F}, \Lambda) \leq \tilde{\Lambda}(\text{Crit}_{\mathcal{F}}(f))$.*

PROOF OF THEOREM 9.9. By Lemma 9.6 and since the critical sets are closed and disjoint, there exists a family of mutually disjoint open sets, $\{U_i\}$ ($i \in \mathbb{N} \cup \{0\}$), where each U_i contains C_i , and such that each U_i is tangentially categorical. We have $\overline{U_i} \subset \tilde{U}_i$, where \tilde{U}_i is an \mathcal{F} -categorical open set that does not contain regular points p with $\omega(p) \in C_j$ for $j > i$.

Let ϕ be the modified gradient flow (Example 9.5); thus $(X \setminus \text{Crit}_{\mathcal{F}}(f), \phi)$ is a 1-dimensional lamination and we can apply Proposition 9.2 and Theorem 9.3 to it.

Let p be a relative minimum ($p \in C_0$), by the continuity of the gradient flow we can find a foliated chart around p where each plaque has at least a relative minimum, and these relative minima converge to p when the plaques approach the plaque of p . Moreover each plaque has only one relative minima if the foliated chart is chosen small enough since, if infinitely many of them have more than one relative minima, then there exists a sequence of points in $C_1(f)$ converging to p by the mountain pass theorem [20], which contradicts the assumption on the critical sets to be closed. Therefore $C_0(f)$ is an (embedded) open transversal meeting each leaf in a discrete set. So U_0 can be chosen to be a tube around C_0 ; in particular, it tangentially contracts to C_0 .

Let $U'_0 = \bigcup_{n \in \mathbb{N}} \phi_{-n}(U_0)$, which is open since $C_0(f)$ consists of relative minima, and it is tangentially categorical since the flow ϕ contracts all the points of U'_0 to C_0 . The set $X_1 = X \setminus U'_0$ is closed. The critical set $C_1(f)$ equals the set of relative minima of the restriction $f|_{X_1}$. Notice that X_1 consists of critical points outside of $C_0(f)$ and regular points connecting these critical points according to the definition of the relation “ \ll ”. Let $F_1 = U_1 \cap X_1$ and $F'_1 = \bigcup_{n \in \mathbb{N}} \phi_{-n}(F_1)$. The set F'_1 is open in X_1 and closed in X . The set $F'_1 \setminus C_1(f)$ is ϕ -saturated and closed in $X \setminus \text{Crit}_{\mathcal{F}}(f)$. There is an open set U'_1 containing F'_1 such that there exists a measurable deformation H satisfying $H(U'_1 \times \{1\}) \subset \tilde{U}_1$.

The set U'_1 is defined as follows. Observe that the flow lines of $(X \setminus \text{Crit}_{\mathcal{F}}(f), \phi)$ are embedded (and not only immersed) in \mathbb{E} by the properties of the gradient flow. We consider $F'_1 \setminus C_1(f)$ as a subset of $X \setminus \text{Crit}_{\mathcal{F}}(f)$ and embedded in the Hilbert space \mathbb{E} . Consider the open subset $V_1 = \bigcup_{\gamma} W(\gamma)$ of $X \setminus \text{Crit}_{\mathcal{F}}(f)$, where γ runs in the family of flow lines in $F'_1 \setminus C_1$ and $W(\gamma)$ is a tubular neighborhood of γ provided by Theorem 9.3. By the Lindelöf property, we can assume that V_1 is a countable union of tubular neighborhoods of flow lines: $V_1 = \bigcup_{n \in \mathbb{N}} W(\gamma_n)$. Let $\pi_n : W(\gamma_n) \rightarrow \gamma_n$ be the projections given by the same Theorem 9.3. We can suppose also that

the family $(\gamma_n)_{n \in \mathbb{N}}$ is locally finite by paracompactness. Let $\lambda_n : V_1 \rightarrow [0, 1]$, $n \in \mathbb{N}$, be a partition of unity associated to the $\{W(\gamma_n)\}_{n \in \mathbb{N}}$. For each $x \in V_1$, let $I(x) \subset \mathbb{N}$ be the set of numbers n such that $x \in W(\gamma_n)$. The isotopy $\phi|_{F'_1}$ contracts F'_1 to C_1 . We extend the deformation $\phi|_{F'_1 \setminus C_1(f)}$ to the neighborhood V_1 . This extension can be defined if we consider our embedding of $X \setminus \text{Crit}_{\mathcal{F}}(f)$ in \mathbb{E} : for $x \in V_1$, $t \in \mathbb{R}$ and $n \in I(x)$, let $r(x, t, n)$ be the unique positive real number such that $\phi_{r(x, t, n)}(x) = \gamma(x) \cap \pi_n^{-1}(\phi_t(\pi_n(x)))$. Let $s : V_1 \times \mathbb{R} \rightarrow X$ be the continuous map defined by $H(x, t) = \phi_{s(x, t)}(x)$, where

$$s(x, t) = \sum_{k \in I(x)} \lambda_k(x) r(x, t, k).$$

For $x \in V_1$ and $t \in \mathbb{R}$, there exists $k_1, k_0 \in I(x)$ such that $r(x, t, k_1) \leq s(x, t) \leq r(x, t, k_0)$. It is clear that $\lim_{t \rightarrow \infty} \phi_{r(x, t, n)} \subset \overline{U_1} \subset \widetilde{U_1}$ for all $n \in \mathbb{N}$. Let $p \in C_1$ and let $x \in F'_1 \setminus C_1$ with $\omega(x) = p$. By the continuity of ϕ , there exists a neighborhood $U(p)$ of $p \cup \gamma(x)$, a tubular neighborhood $V(\gamma(x))$ of $\gamma(x)$ contained in V_1 and $T \in \mathbb{R}$ such that $\phi_{r(y, t, n)} \subset U(p)$ for all $y \in V(\gamma(x))$, $t > T$ and $n \in I(y)$. Therefore $\lim_{t \rightarrow \infty} H(x, t) \subset \overline{\bigcup_{p \in C_1} U(p)} \subset \overline{U_1} \subset \widetilde{U_1}$ for all $x \in \bigcup_{\gamma \in F'_1 \setminus C_1} V(\gamma) \subset V_1$. Then the open subset $V'_1 = \bigcup_{\gamma \in F'_1 \setminus C_1} V(\gamma) \subset X$ is \mathcal{F} -categorical (by a standard change of parameter). Finally, if $\widetilde{U_1}$ is small enough, $U'_1 = V'_1 \cup \widetilde{U_1}$ is \mathcal{F} -categorical by a telescopic argument [17] and $F'_1 \subset U'_1$.

This process can be done inductively by taking $M_n = M \setminus (U'_0 \cup \bigcup_{i=1}^{n-1} F'_i)$, and by using the same trick to define U'_n observing that $C_n(f)$ is the set of relative minima of M_n . \square

PROOF OF THEOREM 9.10. We have assumed that the critical sets meet Λ -almost every leaf in a discrete set. Hence, without loss of generality, we can suppose that leafwise critical points are leafwise isolated. Take U_i like in the proof of Theorem 9.9 so that $\text{Cat}(U_i, \mathcal{F}, \Lambda) \leq \Lambda(C_i) + \varepsilon/2^i$ (by Lemma 9.6). Clearly, defining U'_i in the same way, it is also true that $\text{Cat}(U'_i, \mathcal{F}, \Lambda) \leq \Lambda(C_i) + \varepsilon/2^i$, and the proof is complete. \square

QUESTION 9.11. We can ask if the same results are also true when the critical sets are not closed. We are greatly convinced that the answer is affirmative, but the proof seems to be much more difficult.

Part 3

Secondary Λ -category

CHAPTER 10

Secondary Λ -category

In this chapter, we introduce a refinement of the definition of the Λ -category. The Λ -category is zero in many interesting cases, as follows from the dimensional upper bound (see Corollary 7.21). Now, we focus on the rate of convergence to zero of the expression that defines the Λ -category; this rate will be called the *secondary Λ -category*.

1. Definition and first properties

Let (X, \mathcal{F}) be a lamination on a compact Polish space. For a fixed finite regular atlas \mathcal{U} of \mathcal{F} , recall that a *chain of plaques* of \mathcal{U} is a finite sequence of plaques of charts in \mathcal{U} such that each plaque meets the next one. The *length* of a chain of plaques is its number of elements plus one. For $x, y \in M$ in the same leaf, let $d_{\mathcal{U}}(x, y)$ be the minimum length of a chain of plaques of \mathcal{U} such that the first and last ones contain x and y , respectively. If x and y are points in different leaves, we set $d_{\mathcal{U}}(x, y) = \infty$. This defines a function $d_{\mathcal{U}} : M \times M \rightarrow \mathbb{N} \cup \{\infty\}$ that is symmetric and satisfies the triangle inequality (it is called a *coarse metric* in [19]). Moreover $d_{\mathcal{U}}(x, y) < \infty$ if and only if x and y are in the same leaf, and $d_{\mathcal{U}}(x, y) = 1$ if and only if there is some plaque containing x and y . For $x \in M$ and $X \subset M$, let $d_{\mathcal{U}}(x, X) = \min_{y \in X} d_{\mathcal{U}}(x, y)$. The *length* relative to \mathcal{U} of a path σ on the leaves, denoted by $\text{length}(\sigma)$ (or more explicitly, $\text{length}_{\mathcal{U}}(\sigma)$), is the minimum length of a chain of plaques in \mathcal{U} covering σ . When \mathcal{F} is C^1 , we can fix a Riemannian metric g on the leaves that varies continuously in the transverse direction. Then, if σ is Lipschitz, its *length* $\text{length}(\sigma)$ can be also given by g ; in this case, the notation $\text{length}_g(\sigma)$ can be also used. From the compactness of X , it follows that there is some $A \geq 1$ and $B \geq 0$ such that

$$\frac{1}{A} \text{length}_g(\sigma) - B \leq \text{length}_{\mathcal{U}}(\sigma) \leq A \text{length}_g(\sigma) + B$$

for all σ . Because of this, both definitions of length will work equally well for our purposes. Then we preferably use the length given by a regular atlas since it is defined with more generality. The *length* of a tangential deformation is the supremum of the lengths of the induced leafwise paths; if the length is defined by a leafwise Riemannian metric, we assume that the deformation is Lipschitz.

Let Λ be a transverse invariant measure of \mathcal{F} such that $\text{Cat}(\mathcal{F}, \Lambda) = 0$. In the rest of this section, we assume that Λ is regular and finite on compact sets.

DEFINITION 10.1. Let $\text{Cat}(\mathcal{F}, \Lambda, r)$ be defined like the Λ -category by using only tangential deformations of length $\leq r$. If we use a finite covering \mathcal{U} to define the length, then the more explicit notation $\text{Cat}(\mathcal{F}, \Lambda, r, \mathcal{U})$ can

be used. If we use a leafwise Riemannian metric g to define the length, then we consider only Lipschitz deformations in the definition of $\text{Cat}(\mathcal{F}, \Lambda, r)$, and the more explicit notation $\text{Cat}(\mathcal{F}, \Lambda, r, g)$ can be used.

Observe that $\text{Cat}(\mathcal{F}, \Lambda, n, \mathcal{U})$ is a non-increasing sequence of positive numbers that converges to $\text{Cat}(\mathcal{F}, \Lambda) = 0$.

REMARK 36. It is also true for $\text{Cat}(\mathcal{F}, \Lambda, n)$ that a null transverse set is unessential for its computation. Any σ -compact null transverse set can be covered by tangentially contractible open sets contained in charts such that each chart has a contraction of $\text{length}_{\mathcal{U}} \leq 1$, and the sum of the measures of the final deformations are arbitrarily small (see Proposition 7.7).

DEFINITION 10.2 (Growth types). Let \mathcal{I} be the set of non-negative non-decreasing sequences.

$$\mathcal{I} = \{ g : \mathbb{N} \rightarrow [0, \infty) \mid g(n) \leq g(n+1) \ \forall n \in \mathbb{N} \}.$$

A preorder “ \preceq ” on \mathcal{I} is defined by setting $g \preceq h$ if $\exists B \in \mathbb{N}$ and $\exists A > 0$ so that $g(n) \leq Ah(Bn) \ \forall n \in \mathbb{N}$. This preorder \preceq induces an equivalence relation “ \simeq ” in \mathcal{I} defined by $g \simeq h$ if $g \preceq h \preceq g$. The elements of the quotient set $\mathcal{E} = \mathcal{I} / \simeq$ are called *growth types of non-decreasing sequences* and has the partial order “ \leq ” induced by “ \preceq ”. The equivalence class of g is denoted by $[g]$.

Notice that the sequence $1/\text{Cat}(\mathcal{F}, \Lambda, n, \mathcal{U})$ is a non-decreasing.

DEFINITION 10.3. The growth type of $1/\text{Cat}(\mathcal{F}, \Lambda, n, \mathcal{U})$ is called the *secondary Λ -category* of \mathcal{F} , and is denoted by $\text{Cat}_{\text{II}}(\mathcal{F}, \Lambda, \mathcal{U})$.

PROPOSITION 10.4. *The secondary Λ -category is independent on the choice of \mathcal{U} .*

PROOF. Let \mathcal{V} be another finite regular atlas of \mathcal{F} . There exists some integer $C \geq 1$ such that each chart of \mathcal{U} is covered by at most C charts of \mathcal{V} . Hence any chain of charts of \mathcal{U} of length $n \in \mathbb{N}$ can be covered by a chain of charts of \mathcal{V} of length $\leq Cn$, giving $\text{length}_{\mathcal{U}}(\sigma) \leq C \text{length}_{\mathcal{V}}(\sigma)$ for any leafwise path σ . Finally, any tangential homotopy follows, locally, a chain of charts [33]. Hence any tangential homotopy of $\text{length}_{\mathcal{U}} \leq n$ is of $\text{length}_{\mathcal{V}} \leq Cn$, obtaining $\text{Cat}(\mathcal{F}, \Lambda, Cn, \mathcal{V}) \leq \text{Cat}(\mathcal{F}, \Lambda, n, \mathcal{U})$ for all n , and therefore $\text{Cat}_{\text{II}}(\mathcal{F}, \Lambda, \mathcal{U}) \leq \text{Cat}_{\text{II}}(\mathcal{F}, \Lambda, \mathcal{V})$. The reverse inequality follows with the same argument. \square

According to Proposition 10.4, the secondary Λ -category is denoted by $\text{Cat}_{\text{II}}(\mathcal{F}, \Lambda)$ from now on.

PROPOSITION 10.5. *The secondary Λ -category is invariant by measure preserving tangential homotopy equivalences.*

PROOF. Let (X, \mathcal{F}) and (Y, \mathcal{G}) be compact laminated spaces with a transverse invariant measure, let $h : (X, \mathcal{F}) \rightarrow (Y, \mathcal{G})$ be a measure preserving tangential homotopy equivalence, and let f be a tangentially homotopic inverse of h . We use the same notation Λ for the transverse invariant measures of \mathcal{F} and \mathcal{G} , meaning that h preserves them. If V is a tangentially categorical open set for \mathcal{G} , and $H : V \times [0, 1] \rightarrow \mathcal{G}$ is its tangential contraction, then $h^{-1}(V)$

is tangentially categorical for \mathcal{F} and there exists $K : h^{-1}(V) \times [0, 1] \rightarrow \mathcal{F}$ such that $\Lambda(K(h^{-1}(V) \times \{1\})) \leq \Lambda(H(V \times \{1\}))$ by Proposition 11.4. Recall that the homotopy K is given by

$$K(x, t) = \begin{cases} F(x, 2t) & \text{if } t \leq 1/2 \\ G(x, 2t - 1) & \text{if } t \geq 1/2, \end{cases}$$

where F is the tangential homotopy from the identity map on \mathcal{F} to $f \circ h$, and

$$G : h^{-1}(V) \times [0, 1] \xrightarrow{h \times \text{id}} V \times [0, 1] \xrightarrow{H} Y \xrightarrow{f} X.$$

Since the homotopy F is defined on a compact space, its length is finite; say $\leq C$. Choose foliated regular atlases \mathcal{U} and \mathcal{V} for \mathcal{F} and \mathcal{G} , respectively. By arguing as above, there exists some $D \geq 1$ such that the length of G is $\leq Dn$, where n is the length of H , and D is independent of n . Therefore $\text{Cat}(\mathcal{F}, \Lambda, Dn+C, \mathcal{U}) \leq \text{Cat}(\mathcal{G}, \Lambda, n, \mathcal{V})$, obtaining $\text{Cat}_{\text{II}}(\mathcal{G}, \Lambda) \leq \text{Cat}_{\text{II}}(\mathcal{F}, \Lambda)$. The reverse inequality has an analogous proof. \square

Let T be a complete transversal of \mathcal{F} . Define $\text{Cat}(\mathcal{F}, \Lambda, n, \mathcal{U}, T)$ in the same way as $\text{Cat}(\mathcal{F}, \Lambda, n, \mathcal{U})$ by taking only tangential contractions $H : U \times [0, 1] \rightarrow \mathcal{F}$ such that $H(U \times \{1\}) \subset T$.

PROPOSITION 10.6. *The growth types of*

$$\frac{1}{\text{Cat}(\mathcal{F}, \Lambda, n, \mathcal{U})} \quad \text{and} \quad \frac{1}{\text{Cat}(\mathcal{F}, \Lambda, n, \mathcal{U}, T)}$$

are equal.

PROOF. By using the argument of Proposition 7.19, any tangentially contractible open set U has a partition V_1, \dots, V_k, F , where the sets V_i are open, $\bigcup_i V_i$ contracts to a transversal contained in T , and F is a null-transverse set. By compactness, since T is complete, the length of a deformation of $\bigcup_i V_i$ needed to reach T is bounded by a constant independent of U : the ambient space can be covered by a sequence of saturations of T in the charts of a regular foliated atlas; by compactness, there is a finite number of them, and this number is the desired constant. By invariance, the measure of the final step of this deformation is controlled by the measure of a contraction of U . On the other hand the contribution of F can be made as small as desired by using contractions of length 1 (in the charts of the atlas). \square

2. Secondary category and growth of the holonomy pseudogroup

Now, we give a relation between secondary Λ -category and the growth type of the holonomy group in the case of free suspensions. This result shows the deep relation between the concepts of holonomy and tangential homotopy. The growth rate of the group give us a clear lower bound for the secondary Λ -category. The converse is true with suitable conditions given by the Rohlin tower theorem.

DEFINITION 10.7 (Growth of a group). Let G be a finitely generated group and let S be a symmetric set of generators (here, “symmetric” means that the identity element and the inverse of any element of S belong to

S). For each $g \in G$, let $l_S(g)$ be the minimum number $n \in \mathbb{N}$ such that g can be expressed by the composition of at most n elements of S . Set $S_n = \{g \in G \mid l_S(g) \leq n\}$. The *growth* of G is the growth type of the sequence $\#S_n$, $\text{growth}(G) = [\#S_n]$.

The growth of a finitely generated group is independent of the choice of the finite set of generators.

We give a version of the notion of growth for pseudogroups as follows. In the case of pseudogroups, we have the operations of composition, inversion, restriction to open sets and combination. A set of generators is a set of elements of the pseudogroup such that any other transformation can be obtained from them by using the above operations. We assume that the set is symmetric in the sense that it contains the identity maps on the domains and images of its elements, and is closed by inversion; thus inversion can be removed from the above operations.

DEFINITION 10.8 (Growth of a pseudogroup [36]). Suppose that Γ is finitely generated. Choose a finite symmetric set of generators S and let S_n be the set of transformations obtained by a composition of at most n elements of S ; we assume that S_0 consists of the global identity map. Finally we define $\text{growth}_S(\Gamma) = [\#S_n]$.

Let Λ be a probability measure on T invariant by Γ . Let $K \subset T$ be the support of Λ , which is a Γ -invariant closed subset of T . Then Γ induces a pseudogroup Γ_Λ on K , and a symmetric set S of generators of Γ induces a symmetric set of generators S^Λ of Γ_Λ . The Λ -*growth* of Γ associated to S is $\text{growth}_{S,\Lambda}(\Gamma) = \text{growth}_{S^\Lambda}(\Gamma_\Lambda)$.

The definition of growth of pseudogroups depends on the choice of the set of generators but it is related with the secondary category by the following proposition.

PROPOSITION 10.9. $\text{Cat}_{\text{II}}(\mathcal{F}, \Lambda) \leq \text{growth}_{S^{\mathcal{U}}, \Lambda}(\Gamma)$, where $S^{\mathcal{U}}$ is the finite symmetric system of generators defined by a regular foliated atlas \mathcal{U} .

PROOF. Since $\text{Cat}(\mathcal{F}, \Lambda) = 0$, we can suppose that $\Lambda(T) < \infty$, or even $\Lambda(T) = 1$, where T is a disjoint union of transversals associated to charts of a regular finite foliated atlas \mathcal{U} . In order to compute the secondary Λ -category, we use \mathcal{U} to define the length of tangential homotopies, and Proposition 10.6 to consider only tangential contractions finishing in T .

A finite family of holonomy transformations can be associated to any tangential contraction of finite length since, locally, a tangential contraction follows a chain of charts [33]. For each $n \in \mathbb{N}$, let $H : U \times [0, 1] \rightarrow \mathcal{F}$ be a tangential contraction such that $U \cap \text{supp}_T(\Lambda) \neq \emptyset$ and $\text{length}(H) \leq n$. It is clear that each holonomy transformation h locally defined by H on $\text{supp}_T(\Lambda)$ is a restriction of elements in $S_n^{\mathcal{U}, \Lambda}$. Given any $\varepsilon > 0$, let U_1, \dots, U_N be tangentially categorical open sets covering the ambient space, and let H^1, \dots, H^N be respective tangential contractions of length $\leq n$ such that

$$\sum_i \Lambda(H^i(U_i \times \{1\})) < \text{Cat}(\mathcal{F}, \Lambda, n, \mathcal{U}, T) + \varepsilon.$$

Hence $\{U_i \cap \text{supp}_T(\Lambda)\}_{i=1}^N$ is a covering of $\text{supp}_T(\Lambda)$, and each H^i defines a finite number of local holonomy maps obtained from $S_n^{\mathcal{U}, \Lambda}$ by restriction. Let $\{h_1, \dots, h_K\}$ be the set of such holonomy maps, and let $S_n^{\mathcal{U}, \Lambda} = \{g_1, \dots, g_M\}$, where $M = \#S_n^{\mathcal{U}, \Lambda}$. We can suppose that each U_i has a partition $\{U_{i,j}, F_i\}_{1 \leq j \leq N_i}$, where F_i is a null transverse set, and the sets $U_{i,j}$ are open and so that each $U_{i,j} \cap \text{supp}_T(\Lambda)$ is contained in the dominion of some h_k (see Proposition 7.19). For $1 \leq i \leq M$, let F_i denote the family of maps h_j obtained from g_i by restriction. All of the maps in F_i can be combined to give another holonomy transformation q_i which is a restriction of g_i ; thus all the holonomy maps in F_i are restrictions of q_i . Since the domains of the maps q_i cover $\text{supp}_T(\Lambda)$, it follows that there is some $i_0 \in \{1, \dots, M\}$ such that $\Lambda(\text{dom}(q_{i_0})) \geq 1/M$. By the invariance of the measure,

$$\frac{1}{M} \leq \Lambda(\text{Im}(q_{i_0})) \leq \sum_i \Lambda(H^i(U_i \times \{1\})) < \text{Cat}(\mathcal{F}, \Lambda, n, \mathcal{U}, T) + \varepsilon.$$

The proof follows by taking $\varepsilon > 0$ arbitrarily small. \square

REMARK 37. In the case of foliations given by suspensions of free actions of finitely generated groups on a locally compact Polish space, we can take the bound given by the growth of the group.

3. Case of suspensions with Rohlin groups

In this section, we see a family of examples where the inequality of Proposition 10.9 becomes an equality. They are the cases where the pseudogroup is generated by a free action of a Rohlin group.

DEFINITION 10.10 (Rohlin towers, Rohlin sets and Rohlin groups). Let G be a locally compact, second countable and Hausdorff topological group acting on a standard Borel space X , let Λ be an invariant probability measure on X , and let $F \subset G$ be a Borel subset. We say that a Borel set $V \subset X$ is an F -base in X if $FV = \bigcup_{f \in F} f(V)$ is Borel, $\Lambda(FV) > 0$, and the sets fV ($f \in F$) are disjoint from each other. The set FV in the previous definition is called an F -tower.

A relatively compact set $F \subset G$ is called a *Rohlin set* if, for any free action of G on a standard Borel space X with an invariant probability measure Λ and for any $\varepsilon > 0$, there exists an F -tower $FV \subset X$ with $\Lambda(FV) > 1 - \varepsilon$.

A topological group G as above is called a *Rohlin group* if, for any compact subset $K \subset G$ and for any $\varepsilon > 0$, there exists a Rohlin set $F \subset G$ so that $K \subset F$.

THEOREM 10.11 (C. Series [32]). *Any locally compact, Hausdorff, second countable, almost connected and amenable group is a Rohlin group.*

PROPOSITION 10.12 (Open approximation to a Rohlin tower). *Let G be a discrete Rohlin group acting freely on a locally compact Polish space T with a regular probability invariant measure Λ . Let F be a Rohlin set, and let $\varepsilon > 0$. Then there exists a sequence $\{V_k\}_{k \in \mathbb{N}}$ of open F -bases such that $\sum_k \Lambda(FV_k) > 1 - \varepsilon$.*

PROOF. Any Rohlin set F is finite since G is discrete and F is relatively compact by definition. Let B be an F -base. By regularity of Λ , there exists an open set $V \supset B$ such that $|\Lambda(V) - \Lambda(B)| < \varepsilon/\#F$. By continuity, finiteness of F and since the action is free, there is a partition $\{C, V_1, V_2, \dots\}$ of V such that C is closed on V , each V_i is an open F -base and $\Lambda(C) = 0$ (C is the union of the boundaries of the sets V_i). Finally, by the invariance of Λ ,

$$1 - \varepsilon < \Lambda(FB) \leq \sum_k \Lambda(FV_k) \leq \#F \cdot \left(\Lambda(B) + \frac{\varepsilon}{\#F} \right) \leq 1. \quad \square$$

PROPOSITION 10.13. *Let M and F be compact manifolds, and let $h : \pi_1(M) \rightarrow \text{Homeo}(F)$ be a homomorphism. Suppose that $h(\pi_1(M))$ is a Rohlin group, the induced action of $\pi_1(M)$ on F is free, and there exists an invariant regular probability measure Λ on F . Then $\text{Cat}_{\text{II}}(\widetilde{M} \times_h F, \Lambda) = \text{growth}(h(\pi_1(M)))$.*

PROOF. The inequality $\text{Cat}_{\text{II}}(\widetilde{M} \times_h F, \Lambda) \leq \text{growth}(h(\pi_1(M)))$ holds by Proposition 10.9. Now, let U_1, \dots, U_N be a covering of M by contractible open sets and let $p : \widetilde{M} \times_h F \rightarrow M$ be the projection from the suspension to the base space. We have $p^{-1}(U_i) \approx U_i \times F$ for all i , obtaining a regular foliated atlas \mathcal{U} for the computation of the secondary Λ -category. Let $\varepsilon > 0$ and let S be a symmetric set of generators of $h(\pi_1(M))$. By the Proposition 10.12, for each $n \in \mathbb{N}$, there exists a sequence V_k^n of mutually disjoint open S_n -bases (in F) such that $1 \geq \sum_k \Lambda(S_n V_k^n) > 1 - \varepsilon$. Moreover $\Lambda(\bigcup_k V_k^n) \leq 1/\#S_n$ by the invariance of Λ . By the regularity of Λ , there is a compact subset $K_n \subset \bigcup_k S_n V_k^n$ such that $1 - 2\varepsilon < \Lambda(K_n) < 1 - \varepsilon$; clearly, $\Lambda(F \setminus K_n) < 2\varepsilon$. For each k , consider the saturation $\text{sat}_i(S_n V_k^n)$ of $S_n V_k^n$ on each $p^{-1}(U_i) \approx U_i \times F$; these sets are open in the ambient space and they are clearly tangentially contractible. The contractibility of U_i induces a tangential contraction of $\text{sat}_i(S_n V_k^n)$ to the transversal $S_n V_k^n$, which in turn contracts to V_k^n by a homotopy induced by the holonomy maps in S_n . Thus, by compactness, there exists a constant K independent of n such that all of these induced homotopies have length $\leq Kn$. On the other hand, the saturations $\text{sat}_i(F \setminus K)$ in each $p^{-1}(U_i)$ contract to $F \setminus K$.

Therefore, for each $n \in \mathbb{N}$, the family

$$\{\text{sat}_i(F \setminus K), \text{sat}_i(S_n V_k^n) \mid i \in \{1, \dots, N\}, k \in \mathbb{N}\}$$

is a covering of $\widetilde{M} \times_h F$ by tangentially contractible open sets, giving

$$\text{Cat}(\mathcal{F}, \Lambda, \mathcal{U}, Kn) \leq 4N\varepsilon + \Lambda\left(\bigcup_k V_k^n\right) \leq 4N\varepsilon + \frac{N}{\#S_n}. \quad \square$$

EXAMPLE 10.14. The secondary Λ -category of a minimal foliation by hyperplanes on the n -dimensional torus with the Lebesgue measure in a transverse circle is $[1, 2^n, 3^n, \dots]$. Thus the secondary Λ -category distinguishes the dimension of the ambient manifold of these foliations. Hence the secondary Λ -category gives some new interesting information.

REMARK 38. The upper bound of Proposition 10.9 is an equality in the case of Rohlin suspensions (Proposition 10.13), but not in general as shown

by the following simple example. Let (T^3, \mathcal{F}_1) be the foliation by dense cylinders and (T^3, \mathcal{F}_2) a minimal foliation by hyperplanes. Consider the usual Lebesgue invariant measures in a transverse circle in both cases. Let $\mathcal{F} = \mathcal{F}_1 \sqcup \mathcal{F}_2$ on $T^3 \sqcup T^3$. We have $\text{Cat}(\mathcal{F}, \Lambda, k) \sim \frac{1}{k} + \frac{1}{k^2}$ and $\text{Cat}_{\text{II}}(\mathcal{F}, \Lambda) = [1, 2, 3, \dots]$. However, $\text{growth}_{\Lambda}(\Gamma_{\mathcal{F}_1 \sqcup \mathcal{F}_2}) = \text{growth}_{\Lambda}(\Gamma_{\mathcal{F}_2}) = [1, 2^2, 3^2, \dots]$.

QUESTION 10.15. Is the upper bound of Proposition 10.9 an equality when the transverse invariant measure is ergodic?

4. Pseudogroup invariance

We have seen that the nullity or the positivity of the Λ -category is an invariant of the holonomy pseudogroup and the invariant measure (Corollary 7.32), and the secondary Λ -category is defined in the case of zero Λ -category. The aim of this section is to show the same kind of invariance for the secondary category. Consider the definitions and notation of Section 4.

DEFINITION 10.16. Let Γ be a finitely generated pseudogroup of local transformations of a locally compact Polish space T . Let Λ be a transverse invariant measure of Γ and suppose that $\text{Cat}(\Gamma, \Lambda) = 0$. Let S be a symmetric set of generators of Γ . The *length* of a map $h \in \Gamma$ is the minimum integer k such that h can be locally expressed as a composition of at most k elements in S . Thus, if $h \equiv (h_i)$ is a deformation of a open set $U \subset T$ and each h_i belongs to S_k , then we say that the *length* of the deformation is $\leq k$.

DEFINITION 10.17 (Secondary Λ -category of pseudogroups). With the above hypothesis and notations, let

$$\text{Cat}(\Gamma, \Lambda, S, n) = \inf_{\mathcal{U}, h^U} \sum_{U \in \mathcal{U}} \Lambda(h^U(U)) ,$$

where \mathcal{U} runs in the family of open coverings of T , and, for each $U \in \mathcal{U}$, $h^U \equiv (h_i^U)$ runs in the family of deformations of length $\leq n$ of U . The *secondary Λ -category* of (Γ, S) is $\text{Cat}_{\text{II}}(\Gamma, \Lambda, S) = [1/\text{Cat}(\Gamma, \Lambda, S, n)]$.

DEFINITION 10.18 (Compact generation [16]). Let Γ be a pseudogroup of local transformations of a locally compact space T . It is said that Γ is *compactly generated* if there is a relatively compact open set U in T meeting each orbit of Γ , and such that the restriction \mathcal{H} of Γ to U is generated by a finite symmetric collection $S \subset \mathcal{H}$ so that each $g \in S$ is the restriction of an element \bar{g} of Γ defined on some neighborhood of the closure of $\text{dom}(g)$. The set S is called a system of compact generation of Γ in U .

The holonomy pseudogroup of a lamination on a compact space is compactly generated.

DEFINITION 10.19 ([1]). A finite symmetric family E of generators of a pseudogroup Γ of local transformations of a locally compact space T is said to be *recurrent* if there exists a relatively compact open subset $V \subset T$ and some $R > 0$ such that, for any $x \in Z$, there exists $h \in \Gamma$ with $x \in \text{dom } h$, $\text{length}_E(h) < R$ and $h(x) \in V$. The holonomy maps h will be called *returning maps*.

DEFINITION 10.20. According to the notation in Definitions 10.18 and 10.19, a pseudogroup Γ is called a recurrent compactly generated pseudogroup if Γ is compactly generated and recurrent, and there exists a recurrent system E of Γ such that $V \subset U$, where U and V are like in Definitions 10.18 and 10.19, respectively.

PROPOSITION 10.21. *Let Γ be a recurrent compactly generated pseudogroup in a locally compact Polish space T of finite dimension, let Λ be a Γ -invariant measure, let S be a system of compact generation in U , let \mathcal{H} be the restriction of Γ to U and let E be a recurrent system of generators of Γ on $V \subset U$. Then $\text{Cat}_{\Pi}(\Gamma, \Lambda, E) = \text{Cat}_{\Pi}(\mathcal{H}, \Lambda, S)$.*

PROOF. Since E is recurrent and using the trick of Proposition 7.19, we can suppose that we have a covering $\{U_1, \dots, U_N\}$ of T and deformations $h_i : U_i \rightarrow V$ with $\text{length}_E(h_i) \leq R$. Since U is relatively compact, and any $g \in S$ extends to a map $\bar{g} \in \Gamma$ defined in a neighborhood of \bar{U} , it follows that any map $g \in S$ is a combination of maps in E_K for $K \in \mathbb{N}$ large enough. Using the decomposition trick once more, we obtain that any deformation of $\text{length}_S \leq n$ induces, up to a null transverse set, a deformation of $\text{length} \leq Kn$ relative to E for $K \in \mathbb{N}$ large enough. We can suppose, as usual, that $N = \dim T + 1$, and therefore

$$\text{Cat}(\Gamma, \Lambda, Kn + R, E) \leq (\dim T + 1) \cdot \text{Cat}(\mathcal{H}, \Lambda, S, n),$$

obtaining $\text{Cat}_{\Pi}(\mathcal{H}, \Lambda, S) \leq \text{Cat}_{\Pi}(\Gamma, \Lambda, E)$.

The reverse inequality is easier by using the recurrency in E . There exists a constant K' such that the composition $f \circ h \circ g^{-1}$ of any $h \in E$ with two returning maps, $f : \text{Im } h \rightarrow V \subset U$, $g : \text{dom}_h \rightarrow V \subset U$, of $\text{length} \leq R$ relative to E is of $\text{length} \leq K'$ relative to S for K' large enough. Therefore $\text{Cat}(\mathcal{H}, \Lambda, K'n, S) \leq \text{Cat}(\Gamma, \Lambda, n, E)$ since, locally,

$$h_{i_n} \circ \dots \circ h_{i_1} = f_{i_n} \circ h_{i_n} \circ g_{i_n}^{-1} \circ \dots \circ f_{i_1} \circ h_{i_1} \circ g_{i_1}^{-1},$$

where f_{i_j} and g_{i_j} are returning maps of $\text{length} \leq R$ relative to E . \square

COROLLARY 10.22. *The secondary Λ -category of recurrent compactly generated pseudogroups is independent of the choice of the recurrent set of generators.*

PROOF. If E and E' are two different recurrent systems of generators for Γ relative to open subsets $V \subset U$ and $V' \subset U'$, respectively, where U and U' are relative compact sets associated to the systems of compact generation S and S' , respectively, and let \mathcal{H} and \mathcal{H}' be the restrictions of Γ to U and U' , respectively. Then $E'' = E \cup E'$ is another recurrent system compatible with $V \subset U$ and $V' \subset U'$. Therefore, by using Proposition 10.21, we obtain $\text{Cat}_{\Pi}(\mathcal{H}, \Lambda, S) = \text{Cat}_{\Pi}(\Gamma, \Lambda, E'') = \text{Cat}_{\Pi}(\mathcal{H}', \Lambda, S')$. By the same argument $\text{Cat}_{\Pi}(\Gamma, \Lambda, E) = \text{Cat}_{\Pi}(\Gamma, \Lambda, E')$. \square

REMARK 39. According to Corollary 10.22, we can remove the recurrent set of generators in the notation of the secondary Λ -category of pseudogroups, using simply $\text{Cat}_{\Pi}(\Gamma, \Lambda)$.

PROPOSITION 10.23. *$\text{Cat}_{\Pi}(\Gamma, \Lambda)$ is an invariant by measure preserving equivalences between recurrent compactly generated pseudogroups with invariant measures.*

PROOF. Let Φ be a measure preserving equivalence between recurrent compactly generated pseudogroups with invariant measures, (Γ, T, Λ) and (Γ', T', Λ') . The pseudogroup Γ_Φ on $T_\Phi = T \sqcup T'$ generated by Φ , Γ and Γ' is recurrent and compactly generated, and the combination of the measures Λ and Λ' , Λ_Φ on T_Φ , is Γ_Φ -invariant. So $\text{Cat}_\Pi(\Gamma, \Lambda) = \text{Cat}_\Pi(\Gamma_\Phi, \Lambda_\Phi) = \text{Cat}_\Pi(\Gamma', \Lambda')$ by Proposition 10.21. \square

REMARK 40. The holonomy pseudogroup given by a regular foliated atlas of a lamination in a compact space is recurrent and compactly generated. Observe that the coarse quasi-isometry type of the orbits is independent of the choice of a recurrent system of compact generation, which was the first reason to introduce the concept of recurrency on compactly generated pseudogroups [1].

PROPOSITION 10.24. *Let $(X, \mathcal{F}, \Lambda)$ be a lamination on a compact space with a transverse invariant measure such that $\text{Cat}(\mathcal{F}, \Lambda) = 0$. Let Γ be the holonomy pseudogroup of \mathcal{F} on a complete transversal T . Then $\text{Cat}_\Pi(\mathcal{F}, \Lambda) = \text{Cat}_\Pi(\Gamma, \Lambda)$.*

PROOF. Let $\mathcal{U} = \{U_1, \dots, U_M\}$ be a regular foliated atlas of \mathcal{F} . The inequality $\text{Cat}_\Pi(\mathcal{F}, \Lambda) \leq \text{Cat}_\Pi(\Gamma, \Lambda)$ is obvious since $\text{Cat}(\Gamma, \Lambda, S^\mathcal{U}, n) \leq \text{Cat}(\mathcal{F}, \Lambda, \mathcal{U}, n, T)$, clearly. By using the dimensional trick (see Proposition 7.22), we can show that $\text{Cat}(\mathcal{F}, \Lambda, \mathcal{U}, n) \leq M(m+1) \text{Cat}(\Gamma, \Lambda, S^\mathcal{U}, n)$, where m is the dimension of X . Let $\{V_i\}_{i \in \mathbb{N}}$ be an open covering of T , a complete transversal associated to the atlas \mathcal{U} . Let $\text{sat}_i(B)$ be the saturation of a set B relative to the chart $U_i \in \mathcal{U}$. Of course, $\{\text{sat}_j(V_i \cap T_j)\}_{i,j}$ is an open covering of X . Now let $D_1, \dots, D_{\dim X + 1}$ be an open covering of X such that $D_i = \bigsqcup_j D_{ij}$, where the D_{ij} are open and mutually disjoint, and the whole collection $\{D_{ij}\}_{i,j}$ is a refinement of $\{\text{sat}_j(V_i \cap T_j)\}_{i,j}$ (by Proposition 7.22). Let $(k, l) = I(D_{ij})$ be the first in dexes relative to the lexicographic order such that $D_{ij} \subset \text{sat}_l(V_k \cap T_l)$. Let W_{ikl} be the union of sets D_{ij} such that $I(D_{ij}) = (k, l)$. Each set W_{ikl} contracts to V_k by a deformation of length 0 (the contraction of the chart U_l to its transversal) and therefore any deformation h_i of length n of V_i induces a tangential deformation H of each W_{ikl} of length n such that $H(W_{ikl} \times \{1\}) \subset h(V_i)$. Hence

$$\sum_{ikl} \Lambda(H(W_{ikl} \times \{1\})) \leq M(m+1) \sum_i \Lambda(h_i(V_i)). \quad \square$$

COROLLARY 10.25. *The secondary Λ -category of compact laminations with transverse invariant measures is a transverse invariant (it only depends on the recurrent compactly generated representatives of the holonomy pseudogroup with the corresponding invariant measure).*

5. Cohomological lower bound

In this section, we give a version of the cohomological lower bound of the classical LS category for the secondary Λ -category. We need a notion of “growth of the cup length” taking into account the transverse invariant measure.

Let (M, \mathcal{F}) be a compact foliated manifold. Fix a finite regular foliated atlas \mathcal{U} of \mathcal{F} . The tangential *penumbra* of *radius* $n \in \mathbb{N}$ of X is the set

$$\text{Pen}_{\mathcal{F}}(X, \mathcal{U}, n) = \{x \in M \mid d_{\mathcal{U}}(x, X) < n\}.$$

If \mathcal{F} is C^1 , we can equivalently use the leafwise distance function defined by any Riemannian metric on the leaves which is smooth on M .

Let Λ be a transverse invariant measure of \mathcal{F} .

DEFINITION 10.26. For some measurable transversal T with $\Lambda(T) < \varepsilon$, the set $\text{Pen}_{\mathcal{F}}(T, \mathcal{U}, n)$ is called an $(n, \varepsilon, \mathcal{U}, \Lambda)$ -set.

From now on in this section, assume that \mathcal{F} is C^∞ and $\text{Cat}(\mathcal{F}, \Lambda) = 0$.

DEFINITION 10.27 (Leafwise de Rham cohomology). Let (M, \mathcal{F}) be a C^∞ foliated manifold. Let $\Omega(M, \mathcal{F}) \subset \Omega(M)$ be the subcomplex defined by

$$\Omega^r(M, \mathcal{F}) = \{\omega \in \Omega(M) \mid \omega(X_1, \dots, X_r) = 0 \ \forall X_i \text{ tangent to the leaves}\}.$$

The *leafwise complex* of \mathcal{F} is the quotient $\Omega(M)/\Omega(M, \mathcal{F})$ with the differential $d_{\mathcal{F}}$ induced by the exterior derivative. Its cohomology, $H^*(\mathcal{F}) = \ker d_{\mathcal{F}} / \text{Im } d_{\mathcal{F}}$, is called the *leafwise cohomology* of \mathcal{F} .

REMARK 41. For a general C^∞ lamination (M, \mathcal{F}) , where the ambient space M may not be a manifold, the *leafwise de Rham complex* consists of sections $\omega : M \rightarrow \bigwedge T\mathcal{F}^*$, C^∞ on leaves with all leafwise derivatives of any order continuous in the ambient space M . The *leafwise exterior derivative* $d_{\mathcal{F}}$ is given by the exterior derivative on the leaves.

DEFINITION 10.28. A leafwise differential form ω is called an $(n, \varepsilon, \mathcal{U}, \Lambda)$ -form if $\text{supp}(\omega)$ is contained in an $(n, \varepsilon, \mathcal{U}, \Lambda)$ -set.

DEFINITION 10.29. Define

$$\text{cuplength}(\mathcal{F}, \Lambda, n, \mathcal{U}) = \sup_{\xi} \sum_{\varepsilon \in \xi} \varepsilon,$$

where ξ runs in the family of finite subsets of \mathbb{R}^+ such that, for each $\varepsilon \in \xi$, there is some leafwise closed form α_ε of positive degree so that, for all $(n, \varepsilon, \mathcal{U}, \Lambda)$ -form β_ε , we have $\bigwedge_{\varepsilon \in \xi} (\alpha_\varepsilon - d_{\mathcal{F}} \beta_\varepsilon) \neq 0$ (even though the exterior product is not commutative, it is commutative up to sign, and therefore this inequality makes sense).

REMARK 42. In Definition 10.29, if $\xi = \emptyset$, we set $\text{cuplength}(\mathcal{F}, n, \mathcal{U}) = 0$.

PROPOSITION 10.30. $\text{cuplength}(\mathcal{F}, \Lambda, n, \mathcal{U}) \leq \text{Cat}(\mathcal{F}, \Lambda, n, \mathcal{U})$.

PROOF. Let $\{U_1, \dots, U_K\}$ be an open covering of M . Given any $\delta > 0$, let F^i be a tangential deformation with length $\leq n$ of each U_i such that $\sum_{i=1}^K \varepsilon_i < \text{Cat}(\mathcal{F}, \Lambda, n, \mathcal{U}) + \delta$, where $\varepsilon_i = \Lambda(F^i(U_i \times \{1\}))$. Since $\text{Cat}(\mathcal{F}, \Lambda) = 0$, we can suppose that the sets U_i are tangentially categorical and the maps F^i are tangential contractions. Hence the leafwise forms supported on each U_i are $(n, \varepsilon_i, \mathcal{U}, \Lambda)$ -forms. Since $\{U_1, \dots, U_K\}$ is a covering of M , a classical cohomological argument shows that, for all leafwise closed forms $\alpha_{\varepsilon_1}, \dots, \alpha_{\varepsilon_K}$, there exists an $(n, \varepsilon_i, \mathcal{U}, \Lambda)$ -form β_{ε_i} for each $i \in \{1, \dots, K\}$ such that $\bigwedge_{i=1}^K (\alpha_{\varepsilon_i} - d_{\mathcal{F}} \beta_{\varepsilon_i}) = 0$. \square

DEFINITION 10.31. The *secondary Λ -cuplength* is

$$\text{cuplength}_{\text{II}}(\mathcal{F}, \Lambda, \mathcal{U}) = \left\lceil \frac{1}{\text{cuplength}(\mathcal{F}, \Lambda, n, \mathcal{U})} \right\rceil.$$

PROPOSITION 10.32. *The secondary Λ -cuplength does not depend on the choice of the regular foliated atlas.*

PROOF. The proof is quite similar to the proof of the independence in the choice of atlas for the definition of the secondary Λ -category. Let \mathcal{U} and \mathcal{V} be finite regular foliated atlases. Let K be the minimum integer so that any plaque of \mathcal{U} can be covered by K plaques of \mathcal{V} . Then any $(n, \varepsilon, \mathcal{U}, \Lambda)$ -form is an $(Kn, \varepsilon, \mathcal{V}, \Lambda)$ -form, and therefore

$$\text{cuplength}(\mathcal{F}, \Lambda, Kn, \mathcal{V}) \leq \text{cuplength}(\mathcal{F}, \Lambda, n, \mathcal{U})$$

for all n , yielding

$$\text{cuplength}_{\text{II}}(\mathcal{F}, \Lambda, \mathcal{U}) \leq \text{cuplength}_{\text{II}}(\mathcal{F}, \Lambda, \mathcal{V}). \quad \square$$

COROLLARY 10.33. *The secondary Λ -cuplength is an upper bound for the secondary Λ -category.*

PROOF. By Proposition 10.30, we have

$$\text{cuplength}(\mathcal{F}, \Lambda, \mathcal{U}, n) \leq \text{Cat}(\mathcal{F}, \Lambda, \mathcal{U}, n)$$

for all n , and therefore $\text{Cat}_{\text{II}}(\mathcal{F}, \Lambda) \leq \text{cuplength}_{\text{II}}(\mathcal{F}, \Lambda)$. \square

6. Lower semicontinuity of the secondary Λ -category

Recall that we have adapted for the Λ -category a result of W. Singhof and E. Vogt [33], which asserts that the tangential category is an upper semicontinuous map defined on the space of C^2 foliations over a C^∞ closed manifold M . Now we show that it also has a version for the secondary category.

We work in the same basic set-up and with the same basic notation as in Section 8: For a C^∞ compact manifold M , we consider the space $\mathcal{W}\text{MeasFol}_p^1(M)$ of C^2 foliations of dimension p on M with a regular transverse invariant measure, endowed with the weak topology.

Remember also that the secondary Λ -category is the growth type of a sequence. The set of such growth types is denoted by \mathcal{E} , and it is endowed with an order “ \leq ”. We shall prove the following theorem.

THEOREM 10.34. *Let $(\mathcal{F}_n, \Lambda_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{W}\text{MeasFol}_p^1(M)$ converging to (\mathcal{F}, Λ) such that $\text{Cat}(\mathcal{F}_n, \Lambda_n) = \text{Cat}(\mathcal{F}, \Lambda) = 0$ for all n . Then there exists $N \in \mathbb{N}$ such that $\text{Cat}_{\text{II}}(\mathcal{F}_n, \Lambda_n) \geq \text{Cat}_{\text{II}}(\mathcal{F}, \Lambda)$ for all $n \geq N$.*

We can use simultaneous local parametrizations (see Definition 8.4) in order to choose regular atlases for the \mathcal{F}_n .

PROPOSITION 10.35. *Let U_1, \dots, U_N be a covering of M by \mathcal{F} -categorical regular open sets. Let H^i be tangential contractions of each U^i respectively of length $\leq k$. Suppose also that H^i can be extended to an open neighborhood W_i of $\overline{U_i}$ and the extended contraction is of the same length as H^i . Then there exists $N \in \mathbb{N}$ as large as desired and a covering V_1, \dots, V_N of M such*

that $\overline{V_i} \subset U_i$, each V_i is \mathcal{F}_n -categorical for all $n \geq \mathbb{N}$, and there exists an \mathcal{F}_n -contraction $H^{n,i}$ of length $\leq k+1$ for each V_i so that

$$\left| \sum_i \Lambda_n(H^{n,i}(V_i \times \{1\})) - \sum_i \Lambda(H^i(V_i \times \{1\})) \right| < \varepsilon.$$

PROOF. All properties of the statement are consequences of the results given in the Section 8. We only have to check the assertion about the length of the homotopies $H^{n,i}$. These homotopies are obtained by approximations of H^i given by Proposition 8.2, which can be applied when n is large enough. Finally, these approximations are operated in the sense of Definition 7.16 with a contraction to the transversal $H^i(U_i \times \{1\})$ via a contraction of a local parametrization.

Let $\mathcal{U} = \{N_m\}_{m \in \mathbb{N}}$ be the regular foliated atlas of \mathcal{F} and \mathcal{U}_n the regular foliated atlas associated to each \mathcal{F}_n respectively. We supposed that the charts in each \mathcal{U} and \mathcal{U}_n are associated to simultaneous local parametrizations, and then there exists a bijective map $s_n : \mathcal{U} \rightarrow \mathcal{U}_n$ mapping each chart of \mathcal{U} to the corresponding chart in the simultaneous parametrization of \mathcal{F}_n .

Therefore the length of $H^{n,i}$ is at most the length of H^i for $n \geq N$ since, if $(N_{i_0}, \dots, U_{i_k})$ is a chain of charts in \mathcal{U} covering the deformation H^i , then the chain of charts $(s_n(U_{i_0}), \dots, s_n(U_{i_k}))$ in \mathcal{U}_n covers the deformation $H^{n,i}$. Hence the length of each approximation is greater or equal than the length of H^i . The final contraction increases the length of the tangential contraction by one. \square

PROOF OF THEOREM 10.34. By Proposition 10.35 and using the notation of its statement, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\text{Cat}(\mathcal{F}_n, \Lambda, \mathcal{U}_n, k+1) \leq \text{Cat}(\mathcal{F}, \Lambda, \mathcal{U}, k) + \varepsilon$$

for all $n \geq N$, where \mathcal{U}_n and \mathcal{U} are regular atlases given by simultaneous local parametrizations associated to \mathcal{F}_n and \mathcal{F} , respectively. Therefore $\text{Cat}_{\text{II}}(\mathcal{F}, \Lambda) \leq \text{Cat}_{\text{II}}(\mathcal{F}_n, \Lambda_n)$. \square

7. Critical points

Finally, we relate the secondary Λ -category to the leafwise critical point sets of functions. The secondary Λ -category was defined for laminations on compact spaces and the Theorem 9.9 was stated for finite codimension differentiable Hilbert foliations on Hilbert manifolds; thus we restrict this study to the case of usual differentiable foliations on compact manifolds. However it could be generalized to non-compact manifolds, and even Hilbert foliations, but then it would depend strongly on the choice of the regular atlas. We suppose, as usual, the existence of a complete Riemannian metric on the ambient manifold.

Let $(M, \mathcal{F}, \Lambda)$ be a C^2 lamination with a transverse invariant measure on a compact C^∞ manifold. Suppose that $\text{Cat}(\mathcal{F}, \Lambda) = 0$. Let \mathcal{U} be a finite regular foliated atlas. For each $f \in C^2(M)$ with isolated critical points on the leaves, let $\Phi(f)$ denote the leafwise flow associated to the leafwise gradient of f (see Example 9.5).

DEFINITION 10.36. The function f is said to be (n, \mathcal{U}) -bounded if the \mathcal{U} -length of all flow line of $\Phi(f)$ is $\leq n$.

PROPOSITION 10.37. *Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of ω -PS functions on M such that each f_n is (n, \mathcal{U}) -bounded. Then $[1/\Lambda(\text{Crit}_{\mathcal{F}}(f_n))] \leq \text{Cat}_{\text{II}}(\mathcal{F}, \Lambda)$.*

PROOF. By the proof of Proposition 9.10, $\text{Cat}(\mathcal{F}, \Lambda) \leq \Lambda(\text{Crit}_{\mathcal{F}}(f_n))$ for any n . The proof uses deformations given by a modified gradient flow for each f_n , and therefore these deformations are of length $\leq n$ (with respect to \mathcal{U}) since f_n is (n, \mathcal{U}) -bounded (the trace of the flow lines of the gradient flow and the modified gradient flow are the same). Therefore $\text{Cat}(\mathcal{F}, \Lambda, \mathcal{U}, n) \leq \Lambda(\text{Crit}_{\mathcal{F}}(f_n))$. \square

REMARK 43. In examples, it is difficult to verify the property of (n, \mathcal{U}) -boundedness. Alternatively, we may use a leafwise Riemannian metric g , obtaining a similar condition of (n, g) -boundedness (any integral curve has g -length $\leq n$); to check this condition, we can use the geometry of the leaves.

Part 4

Dynamical categories

CHAPTER 11

Dynamical LS category

Now, our target is to find an invariant similar to the (secondary) Λ -category, but suitable for all laminations on compact metric spaces, without requiring the existence of a transverse invariant measure, which is a strong restriction. We also suppose that the dimension of the ambient space is finite. In the following, the diameter of any set B is denoted by $\text{diam}(B)$.

1. Definition of the dynamical category

Let (X, \mathcal{F}) be a lamination on a metric space (X, d) with finite topological dimension. For each open set $U \subset X$, define

$$\tau_d(U) = \inf \{ \text{diam}(H(U \times \{1\})) \mid H \text{ tangential deformation on } U \} .$$

Then the *dynamical category* of (X, \mathcal{F}) is defined as

$$\text{Cat}(\mathcal{F}, d) = \inf_{\mathcal{U}} \sum_{U \in \mathcal{U}} \tau_d(U) ,$$

where \mathcal{U} runs in the coverings of X by open sets.

PROPOSITION 11.1. *If $\text{Cat}(\mathcal{F}, d) = 0$, then*

$$\text{Cat}(\mathcal{F}, d) = \inf_{\mathcal{U}} \sum_{U \in \mathcal{U}} \tau_d(U) ,$$

where \mathcal{U} runs in the family of coverings of X by \mathcal{F} -categorical open sets.

PROOF. Let U_1, \dots, U_N be an open covering of X such that $\sum_{i=1}^N \tau_d(U_i) < \varepsilon$. By a consequence of the theory of topological dimension, there exist open sets $V_j = \bigcup_k V_{jk}$, $j = 1, \dots, \dim X + 1$, such that $V_{jk} \cap V_{jk'} = \emptyset$ for $k \neq k'$ and the family $\{V_{jk}\}_{j,k}$ is a refinement of the covering $\{U_i\}_i$. We can obtain V_{jk} small enough so that they are tangentially categorical. Therefore each V_j is tangentially categorical. Define $W_{ij} = U_i \cap V_j$. Any deformation of U_i can be restricted to each W_{ij} . Hence $\sum_{ij} \tau_d(W_{ij}) < (\dim X + 1) \cdot \varepsilon$. \square

Thus we can restrict our definition to tangentially categorical open sets when the value is zero. The reverse statement is not true, as we show in the following example.

EXAMPLE 11.2. Let (T^3, \mathcal{F}) be laminated as a product $T^2 \times S^1$ with leaves $T^2 \times \{z\}$, $z \in S^1$. Let $i : T^2 \rightarrow \mathbb{R}^3$ be a topological embedding obtained by revolution of the circle along the y -axis:

$$\left\{ (x, y, z) \in \mathbb{R}^3 \mid (x - \frac{5}{4})^2 + y^2 = 1, \ z = 0 \right\} .$$

Consider the metric on T^2 induced by the usual metric of \mathbb{R}^3 . Now consider the usual metrics on T^2 and S^1 and the corresponding taxicab metric, d , on $T^2 \times S^1$. Let

$$\begin{aligned} U_1 &= \{(x, y, z) \in i(T^2) \mid |x| < 2\} \times S^1, \\ U_2 &= \{(x, y, z) \in i(T^2) \mid |x| > 1\} \times S^1. \end{aligned}$$

Of course, U_1 and U_2 cover T^3 and they are not tangentially categorical. However they tangentially deform to the set

$$K = \left\{ (x, y, z) \in i(T^2) \mid x^2 + y^2 = \frac{1}{4}, y = 0 \right\} \times S^1,$$

and $\text{diam}(K) = \frac{5}{2}$. Hence $\text{Cat}(\mathcal{F}, d) \leq 5$ (in fact this is an equality).

We need at least three tangential categorical open sets to cover T^3 since $\text{Cat}(T^2) = 3$. Let $\mathcal{U} = \{U_1, \dots, U_N\}$ be an \mathcal{F} -categorical open covering and let $H^n : U_n \times \mathbb{R} \rightarrow \mathcal{F}$ be tangential contractions. Of course, $T = \bigsqcup_{n=1}^N H^n(U_n \times \{1\})$ is a transversal and any leaf meets this transversal at least in 3 points since the category of any leaf is 3. Since $\{*\} \times S^1$ is a complete transversal with minimal diameter, we obtain that

$$\sum_n \text{diam}(H^n(U_n \times \{1\})) \geq 3 \cdot \text{diam}(\{*\} \times S^1) = 6.$$

Thus $\sum_{U \in \mathcal{U}} \tau_d(U) \geq 6$.

PROPOSITION 11.3. *The positivity or nullity of the dynamical category does not depend on the uniformity type of the metric.*

PROOF. The proof is quite similar to the proof of Proposition 11.1. Observe that the ambient space is compact and we must use the same dimensional trick in order to prove that both categories (relative to different uniformly equivalent metrics) are zero at the same time. \square

Now, we show that the positivity or nullity of dynamical category is a tangential homotopy invariant. We prove this result for Lipschitz tangential homotopy equivalences with some Lipschitz tangential homotopy inverse. More precisely, there exists $C > 0$ such that $d'(h(x), h(y)) \leq C d(x, y)$ and $d(g(x'), g(y')) < C d'(x', y')$ for all $x, y \in X$, $x', y' \in Y$, where $h : X \rightarrow Y$ and $g : Y \rightarrow X$ are the homotopic equivalence and its homotopic inverse, respectively. This kind of equivalence of laminations will be called *Lipschitz homotopy equivalence*.

PROPOSITION 11.4 (The dynamical category is a tangential homotopy invariant). *Let (X, \mathcal{F}, d) and (Y, \mathcal{G}, d') be Lipschitz homotopy equivalent laminations. Then $\text{Cat}(\mathcal{F}, d) = 0$ if and only if $\text{Cat}(\mathcal{G}, d') = 0$.*

PROOF. Suppose $\text{Cat}(\mathcal{G}, d') = 0$. By using Proposition 11.1, there exists an open covering \mathcal{U} by \mathcal{F} -categorical sets such that $\sum_{U \in \mathcal{U}} \tau_{d'}(U) < \varepsilon$, for any given $\varepsilon > 0$. Let $h : X \rightarrow Y$ be a tangential Lipschitz homotopy equivalence and g its tangential homotopy inverse satisfying the hypothesis. The family $\{h^{-1}(U) \mid U \in \mathcal{U}\}$ is a covering of X by open sets. We will prove that $\tau_d(h^{-1}(U)) \leq C \tau_{d'}(U)$ for all $U \in \mathcal{U}$. Let H^U be a tangential deformation of each $U \in \mathcal{U}$ such that $\sum_{U \in \mathcal{U}} \text{diam}(H^U(U \times \{1\})) < \varepsilon$, and let

$F : (X, \mathcal{F}) \times \mathbb{R} \rightarrow (X, \mathcal{F})$ be a tangential homotopy connecting the identity map and $g \circ h$; i.e., $F(x, 0) = x$ and $F(x, 1) = g \circ h(x)$ for all $x \in X$. Then consider the tangential homotopy G^U given by the composite

$$h^{-1}(U) \times \mathbb{R} \xrightarrow{h \times \text{id}} U \times \mathbb{R} \xrightarrow{H^U} Y \xrightarrow{g} X.$$

The tangential homotopy, $K : h^{-1}(U) \times \mathbb{R} \rightarrow X$, defined by

$$K(x, t) = \begin{cases} F(x, 2t) & \text{if } t \leq 1/2 \\ G^U(x, 2t - 1) & \text{if } t \geq 1/2, \end{cases}$$

is a tangential deformation of each $h^{-1}(U)$. By definition, $G^U(x, 1) = g(H^U(h(x), 1))$. Hence

$$\text{diam}(G^U(h^{-1}(U) \times \{1\})) \leq C \text{diam}(H^U(U \times \{1\})),$$

yielding $\tau_d(h^{-1}(U)) \leq C \tau_{d'}(U)$ for all $U \in \mathcal{U}$. Therefore $\text{Cat}(\mathcal{F}, d) \leq C \text{Cat}(\mathcal{G}, d') = 0$. \square

REMARK 44. By Proposition 11.4, the positivity or the nullity of the dynamical category is an invariant of tangential Lipschitz homotopy equivalences and uniform types of metrics. Therefore we only are interested on knowing when the dynamical category is zero or positive. In the following, we denote the dynamical category by $\text{Cat}_{\text{dyn}}(\mathcal{F})$ with no mention to the metric. In this way, for the following study, we fix a metric d on X .

PROPOSITION 11.5 (Dimensional upper bound). *Let (X, \mathcal{F}) be a lamination, and let $\{T_i\}_{i \in \mathbb{N}}$ be a family of local transversals whose union meets all leaves. Then $\text{Cat}(\mathcal{F}, d) \leq (\dim X + 1) \cdot \sum_i \text{diam}(T_i)$.*

PROOF. Let $T = \bigsqcup_i T_i$ be a complete transversal satisfying and let $\mathcal{U} = \{U_n\}_{n \in \mathbb{N}}$ be a regular foliated atlas of \mathcal{F} . Since T is a complete transversal, we can suppose that any transversal associated to each chart of \mathcal{U} is the domain of a holonomy map with image in some T_i . By using the dimensional trick (Proposition 7.22), we can find a family of open sets $V_j = \bigcup_k V_{jk}$, $j = 1, \dots, \dim X + 1$, such that $V_{jk} \cap V_{jk'} = \emptyset$ for $k \neq k'$, and the family $\{V_{jk}\}_{j,k}$ is a refinement of the covering $\{U_n\}$. Let $n(j, k)$ the minimum n such that $V_{jk} \subset U_n$. Each V_{jk} contracts to a transversal via de contraction of $U_{n(j,k)}$ to an associated transversal. By the assumption on \mathcal{U} , this transversal is the domain of a holonomy map with image in some T_i ; let $I(j, k)$ be the minimum index i with this property. Let W_{ij} be the union of the sets V_{jk} such that $I(V_{jk}) = i$, which form a covering of X . By Proposition 7.19, each W_{ij} tangentially contracts into T_i . Therefore

$$\text{Cat}(\mathcal{F}, d) \leq (\dim X + 1) \cdot \sum_i \text{diam}(T_i). \quad \square$$

REMARK 45. For C^2 codimension 1 foliations on compact manifolds, it is also true that $\text{Cat}(\mathcal{F}, d) \leq (\dim \mathcal{F} + 1) \cdot \sum_i \text{diam}(T_i)$. The proof is the same that works for the Λ -category by using Lemma 7.15. For higher codimension, the arguments do not work since the residual sets given by Lemma 7.15 may not be decomposed into transverse sets of small diameter (but, of course, they can be chosen to have null measure relative to a transverse invariant measure finite in compact sets).

COROLLARY 11.6. *The dynamical category of a lamination with all leaves compact on a compact space is positive.*

PROOF. By using the Epstein filtration [14], there exists a countable filtration $X = B_0 \supset B_1 \supset \cdots \supset B_\alpha \supset B_{\alpha+1} \supset \cdots$ such that $B_{\alpha+1}$ is closed in B_α for all α and $B_\alpha \setminus B_{\alpha+1}$ is the set of leaves with trivial holonomy in the lamination B_α . It is defined by transfinite induction: B_1 is the set of leaves with non-trivial holonomy, which is open and dense in X , and B_α is the set of leaves with non-trivial holonomy in $\bigcap_{\beta < \alpha} B_\beta$. This filtration is always countable, even though it may involve infinite ordinals α .

Let $B \subset X$ be the set of leaves with trivial holonomy. For any leaf $L \subset B$, there exists a neighborhood U of L and a local transversal T meeting L such that $T \times L$ is homeomorphic to U . Consider a metric d on X such that (U, d) is isometric to the taxicab metric on $T \times L$. For any countable collection of local transversals T_i whose union meets all leaves, we obtain that $\sum_i \text{diam}_d(T_i \cap U) > \text{diam}_d(T)$. Hence $\text{Cat}_{\text{dyn}}(\mathcal{F}) > 0$ by Corollary 11.7. Since the ambient space is compact, this is true for any metric. \square

COROLLARY 11.7. *$\text{Cat}_{\text{dyn}}(\mathcal{F}) = 0$ if and only if, for all $\varepsilon > 0$, there exists a complete transversal $T = \bigsqcup_{i=1}^\infty T_i$ such that $\sum_i \text{diam}(T_i) < \varepsilon$.*

PROOF. The “only if” part is clear since, by Proposition 11.1, we can use tangentially categorical open sets to compute the dynamical category and the final step of a tangential deformation to a transversal. For the reverse, we only have to use Proposition 11.5. \square

2. The dynamical category is a transverse invariant

Since we are only interested on the positivity or nullity of the dynamical category, we work with the characterization given in Corollary 11.7, and we pass this condition to pseudogroups. We want to show that this condition is a transverse invariant.

DEFINITION 11.8. Let d be a metric on a Polish space of finite dimension, T . The dynamical category of a pseudogroup (T, d, Γ) is defined as

$$\text{Cat}(\Gamma, d) = \inf \sum_i \text{diam}(T_i) .$$

where $\bigcup_i T_i$ meets each Γ -orbit and each T_i is open.

DEFINITION 11.9. Two pseudogroups, (T, d, Γ) and (T', d', Γ') , are called *uniformly bi-Lipschitz* (  tal  ) equivalent if there exists an equivalence of pseudogroups whose maps are bi-Lipschitz with a global bound for their bi-Lipschitz distortions.

PROPOSITION 11.10. *Let (T, d, Γ) and (T', d', Γ') be uniformly bi-Lipschitz equivalent pseudogroups. Then $\text{Cat}(\Gamma, d) = 0$ if and only if $\text{Cat}(\Gamma', d') = 0$.*

PROOF. If $\varepsilon > 0$ is small enough, then each T_i of diameter $< \varepsilon$ is contained in a domain of a map of the equivalence. It follows from the definition of dynamical category that $\text{Cat}(\Gamma', d') \leq K \text{Cat}(\Gamma, d)$ if $\text{Cat}(\Gamma, d)$ is small enough, where K is a global upper-bound for the bi-Lipschitz distortions. The reverse inequality is analogous. \square

DEFINITION 11.11. Let (X, \mathcal{F}, d) be a lamination in a metric space. It will be called *Lipschitz* if all the holonomy maps are Lipschitz.

COROLLARY 11.12. *Let (X, \mathcal{F}, d) be a Lipschitz lamination in a compact metric space of finite dimension. Let T and T' be a complete transversal and let Γ_T and $\Gamma_{T'}$ be the corresponding representatives of the holonomy pseudogroup. Then $\text{Cat}(\Gamma_T, d) = 0$ if and only if $\text{Cat}(\Gamma_{T'}, d) = 0$.*

PROOF. The result follows from the fact that these two pseudogroups are uniformly bi-Lipschitz equivalent. \square

PROPOSITION 11.13. *Let (X, \mathcal{F}, d) be a lamination on a compact metric space of finite dimension. Let T be a complete transversal and let Γ_T be the corresponding representative of the holonomy pseudogroup. Then $\text{Cat}(\Gamma_T, d) = 0$ if and only if $\text{Cat}(\mathcal{F}, d) = 0$.*

PROOF. It is clear that $\text{Cat}(\Gamma_T, d) \leq \text{Cat}(\mathcal{F}, d)$ since tangential homotopies induce, locally, holonomy maps. By Corollary 11.7 and Proposition 11.5, we obtain that $\text{Cat}(\Gamma_T, d) = 0$ implies $\text{Cat}(\mathcal{F}, d) = 0$. \square

Propositions 11.10 and 11.13 mean that, for Lipschitz laminations on compact metric spaces, the nullity or positivity of the dynamical category is an invariant of their holonomy pseudogroups.

REMARK 46. Of course, we could work with a definition of dynamical category of pseudogroups by using Γ -deformable sets. This will be the case for the corresponding secondary invariant.

CHAPTER 12

The secondary dynamical category

In this chapter, we introduce a refinement for the definition of the dynamical category. Our previous invariant is trivial in many interesting cases, as we can deduce from its characterization (Corollary 11.7). Now, we consider the rate of the expression of the definition of dynamical category when it goes to zero, giving rise to the *secondary dynamical category*.

1. Definition and first properties

DEFINITION 12.1. Let (X, \mathcal{F}, d) be a lamination on a compact metric space and let \mathcal{U} be a regular foliated atlas. Suppose $\text{Cat}(\mathcal{F}, d) = 0$. Then define the n -dynamical category category of \mathcal{F} , $\text{Cat}(\mathcal{F}, d, n, \mathcal{U})$, like the dynamical category, but tanking only tangential deformations of length $\leq n$ relative to the atlas \mathcal{U} . Since X is compact, $\lim_{n \rightarrow \infty} \text{Cat}(\mathcal{F}, d, n, \mathcal{U}) = \text{Cat}(\mathcal{F}, d) = 0$. The sequence $1/\text{Cat}(\mathcal{F}, d, n, \mathcal{U})$ is increasing. Its growth type is called the *secondary dynamical category* of \mathcal{F} and is denoted by $\text{Cat}_{\text{dyn}, \text{II}}(\mathcal{F}, d, \mathcal{U})$.

PROPOSITION 12.2. *The secondary dynamical category is independent of the choice of the regular foliated atlas of X .*

PROOF. Since X is compact, that the foliated regular atlases are finite. Let \mathcal{U} and \mathcal{V} be different regular foliated atlases for \mathcal{F} . There exists a number M such that any chain of charts of \mathcal{U} of length $n \in \mathbb{N}$ can be covered by a chain of charts of \mathcal{V} of length $\leq Mn$. The proof of this fact is easy: each chart of \mathcal{U} is covered by a finite number of chain charts of \mathcal{V} , and then choose M equal to the maximum of the lengths of these chains over all the charts of \mathcal{U} . Finally, observe that any tangential homotopy follows, locally, a chain of charts [33]. Hence any tangential homotopy of length $\leq n$ relative to \mathcal{U} is of length $\leq Mn$ relative to \mathcal{V} , obtaining that $\text{Cat}(\mathcal{F}, d, Mn, \mathcal{V}) \leq \text{Cat}(\mathcal{F}, d, n, \mathcal{U})$, yielding $\text{Cat}_{\text{den}, \text{II}}(\mathcal{F}, d, \mathcal{U}) \leq \text{Cat}_{\text{dyn}, \text{II}}(\mathcal{F}, d, \mathcal{V})$. The reverse inequality follows with the same arguments. \square

In the following, the secondary dynamical category will be denoted by $\text{Cat}_{\text{dyn}, \text{II}}(\mathcal{F}, d)$, without reference to the foliated regular atlas.

PROPOSITION 12.3. *The secondary dynamical category is invariant by Lipschitz tangential homotopy equivalences with Lipschitz tangential homotopy inverse.*

PROOF. Let $h : (X, \mathcal{F}, d) \rightarrow (Y, \mathcal{G}, d')$ be a Lipschitz tangential homotopy equivalence and f a Lipschitz tangential homotopy inverse of h . Let $C > 0$ be a distortion for h and f . If V is a \mathcal{G} -categorical open set and $H : V \times \mathbb{R} \rightarrow Y$ is a tangential contraction, then $h^{-1}(V)$ is \mathcal{F} -categorical

and there exists $K : h^{-1}(V) \times \mathbb{R} \rightarrow X$ such that

$$\text{diam}(K(h^{-1}(V)) \times \{1\}) \leq C \text{diam}(H(V \times \{1\}))$$

by Proposition 11.4; remember that the homotopy K is given by

$$K(x, t) = \begin{cases} F(x, 2t) & \text{if } t \leq 1/2 \\ G(x, 2t - 1) & \text{if } t \geq 1/2, \end{cases}$$

where F is the tangential homotopy from $\text{id} : X \rightarrow X$ to $f \circ h$, and G is defined as the composite

$$h^{-1}(V) \times \mathbb{R} \xrightarrow{h \times \text{id}} V \times \mathbb{R} \xrightarrow{H} Y \xrightarrow{f} X.$$

Now, since F is defined on a compact space, its length is bounded by a constant, say B . Choose regular foliated atlases, \mathcal{U} and \mathcal{V} , for \mathcal{F} and \mathcal{G} , respectively. By arguing as above, we get that there exists M such that the length of G is $\leq Mn$, where n is the length of H , with M independent of n . Therefore $\text{Cat}(\mathcal{F}, d, Mn + B, \mathcal{U}) \leq C \text{Cat}(\mathcal{G}, d, n, \mathcal{V})$, obtaining

$$\text{Cat}_{\text{dyn}, \Pi}(\mathcal{G}, d') \leq \text{Cat}_{\text{dyn}, \Pi}(\mathcal{F}, d).$$

The reverse inequality has an analogous proof. \square

Of course, with similar arguments, it follows that the secondary category is independent of the choice of the metric in the same quasi-isometry class.

DEFINITION 12.4. Let (X, \mathcal{F}, d) be a lamination on a compact metric space and let T be a complete transversal. Define $\text{Cat}(\mathcal{F}, d, n, \mathcal{U}, T)$ in the same way as $\text{Cat}(\mathcal{F}, d, n, \mathcal{U})$, but taking only tangential contractions $H : U \times [0, 1] \rightarrow X$ such that $H(U \times \{1\}) \subset T$.

PROPOSITION 12.5. *Let (X, \mathcal{F}, d) be a Lipschitz lamination on a compact metric space and let T be a complete transversal. Then*

$$\left[\frac{1}{\text{Cat}(\mathcal{F}, d, n, \mathcal{U})} \right] = \left[\frac{1}{\text{Cat}(\mathcal{F}, d, n, \mathcal{U}, T)} \right].$$

PROOF. Let U be a tangential categorical open set and H a tangential deformation of U . By a dimensional argument, we can suppose that $U = \bigsqcup_i U_i$ such that $H(U_i \times \{1\})$ is contained in the domain of a holonomy map h^U with image in T . By compactness and since the lamination is Lipschitz, there exist constants $M, K > 0$ such that the length of h^U relative to \mathcal{U} is $\leq M$ and $\text{diam}(h^U(H(U \times \{1\}))) \leq K \text{diam}(H(U \times \{1\}))$, yielding

$$\text{Cat}(\mathcal{F}, d, n + M, \mathcal{U}, T) \leq K(\dim X + 1) \text{Cat}(\mathcal{F}, d, n, \mathcal{U}).$$

The reverse inequality is trivial. \square

2. Dynamical secondary category and growth

Now, we give a relation between the secondary dynamical category and the growth type of the holonomy group in the case of free suspensions. It is analogous to the case of the secondary Λ -category.

PROPOSITION 12.6 (Upper bound for the secondary dynamical category). *Let (X, \mathcal{F}, d) be a Lipschitz lamination on a compact metric space and suppose that $\text{Cat}(\mathcal{F}, d) = 0$. Let \mathcal{U} be a regular foliated atlas. Then*

$$\left\lceil \frac{2^{-n}}{\text{Cat}(\mathcal{F}, d, n, \mathcal{U})} \right\rceil \leq \text{growth}_{\mathcal{S}\mathcal{U}}(\Gamma) ,$$

where Γ is a representative of the holonomy pseudogroup of \mathcal{F} given by the disjoint union of transversals associated to \mathcal{U} , and $S^{\mathcal{U}}$ is the symmetric set of generators of Γ given by the transverse coordinate change of foliated charts.

PROOF. Since X is compact, its diameter is finite. We can take T equal to a disjoint union of transversals associated to the charts of the regular foliated atlas \mathcal{U} . In order to compute the secondary category, we shall use Proposition 10.6 with the given \mathcal{U} to measure the length of tangential homotopies. Thus we only consider tangential contractions finishing in T . We can suppose $\text{diam}(T) = 1$ without loss of generality.

A family of holonomy transformations is associated to any tangential deformation since, locally, it follows a chain of charts. For each $n \in \mathbb{N}$, let $H : U \times [0, 1] \rightarrow \mathcal{F}$ be a tangential contraction such that $U \cap T \neq \emptyset$, $H(U \times \{1\}) \subset T$ and $\text{length}_{\mathcal{U}}(H) \leq n$. It is clear that each holonomy transformation h , locally defined by H , is a restriction of elements in $S_n^{\mathcal{U}}$ (see Definition 10.8). Let U_1, \dots, U_N be tangentially categorical open sets covering X and let H^1, \dots, H^N be the respective tangential contractions of length $\leq n$ such that

$$\sum_i \text{diam}(H^i(U_i \times \{1\})) < \text{Cat}(\mathcal{F}, d, n, \mathcal{U}, T) + \varepsilon$$

for any given $\varepsilon > 0$. Hence $\{U_i \cap T\}_{i=1}^N$ is a covering of T , and each H^i defines a finite number of local holonomy maps obtained from S_n by restriction. Let $\{h_1, \dots, h_K\}$ be the set of such holonomy maps and set $S_n^{\mathcal{U}} = \{g_1, \dots, g_M\}$ where $M = \#S_n^{\mathcal{U}}$. For $1 \leq i \leq M$, set

$$F_i = \{h_j \mid h_j \text{ is obtained by restriction from } g_i\} .$$

All the elements in F_i can be combined to define another holonomy transformation q_i that is a restriction of g_i , and all the holonomy maps in F_i are restrictions of q_i . Since the domains of the maps q_i , $1 \leq i \leq M$, cover T , we get the existence of an index $j \in \{1, \dots, M\}$ such that $\text{diam}(\text{dom}(q_j)) \geq \frac{1}{M}$. Finally, by compactness and the Lipschitz condition, there exists a constant $K > 1$ such that

$$\begin{aligned} \frac{1}{M} &\leq \text{diam}(q_j) \leq K^n \text{diam}(\text{im}(q_j)) \\ &\leq \sum_i K^n \text{diam}(H^i(U_i \times \{1\})) < K^n \text{Cat}(\mathcal{F}, d, n, \mathcal{U}, T) + K^n \varepsilon . \end{aligned}$$

By taking $\varepsilon > 0$ arbitrarily small, we obtain

$$\frac{K^{-n}}{\text{Cat}(\mathcal{F}, d, n\mathcal{U})} \leq \#S_n^{\mathcal{U}} .$$

Finally observe that the growth type of the right term is independent of the choice of $K > 1$. \square

EXAMPLE 12.7. Let $f : [0, 1] \rightarrow [0, 1]$ be defined by $f(x) = (x^2 + x)/2$. Let \mathcal{F} be the suspension of this diffeomorphism, which is a Lipschitz lamination on the cylinder $S^1 \times [0, 1]$ with two compact leaves (the leaves corresponding to 0 and 1 in $[0, 1]$), and the other leaves are spirals converging to the compact ones. A regular foliated atlas can be given by $U_1 \times [0, 1]$ and $U_2 \times [0, 1]$, where U_1 and U_2 is a covering of S^1 by contractible relatively compact open sets. Of course, $\text{growth}_{\mathcal{S}\mathcal{U}}(\Gamma) = [n]$ but it is easy to compute that $\text{Cat}_{\text{dyn}, \text{II}}(\mathcal{F}) = [2^n]$.

Hence, in Proposition 12.6, the term 2^{-n} cannot be removed in general. However, we can achieve a more useful inequality by assuming a global Lipschitz bound for all holonomy map $h \in S_*^{\mathcal{U}} = \bigcup_{n=0}^{\infty} S_n^{\mathcal{U}}$; in this case, it will be said that the lamination (X, \mathcal{F}, d) is *uniformly Lipschitz*. It is easy to check that this condition is independent of the choice of \mathcal{U} . Obviously, a uniformly Lipschitz lamination is equicontinuous [1]. The following result is elementary.

LEMMA 12.8. *For each open subset V of a complete transversal T with $\text{diam}(V) > 0$, let*

$$S_*^{\mathcal{U}}(V) = \{ h \in S_*^{\mathcal{U}} \mid V \subset \text{dom } h \} ,$$

$$B(V) = \inf_{h \in S_*^{\mathcal{U}}(V)} \text{diam } h(V) .$$

Then (X, \mathcal{F}, d) is uniformly Lipschitz if and only if $B(V) > 0$ for all such V and $\frac{\text{diam}(V)}{B(V)}$ is uniformly bounded on V .

PROPOSITION 12.9. *Let (X, \mathcal{F}, d) be a uniformly Lipschitz lamination on a compact metric space with $\text{Cat}(\mathcal{F}, d) = 0$. Then $\text{Cat}_{\text{dyn}, \text{II}}(\mathcal{F}, d) \leq \text{growth}(\Gamma, S^{\mathcal{U}})$, where $S^{\mathcal{U}}$ is the symmetric set of generators of Γ associated to \mathcal{U} .*

PROOF. Using the same notation and assumption of the proof of Proposition 12.6, we only have to see that we can obtain a constant $K > 1$ independent on n such that

$$\frac{1}{\#S_n^{\mathcal{U}}} \leq \text{diam}(\text{dom } q_j) \leq K \text{diam}(\text{im}(q_j))$$

$$\leq \sum_i K \text{diam}(H^i(U_i \times \{1\})) < K \text{Cat}(\mathcal{F}, d, n, \mathcal{U}, T) + K\varepsilon .$$

This is clear since

$$\text{diam}(\text{dom } h) \leq \frac{\text{diam}(\text{dom } h) \cdot \text{diam}(\text{im } h)}{B(\text{dom } h)} .$$

Hence, according to Lemma 12.8, we only have to choose an upper bound K for the numbers $\frac{\text{diam}(V)}{B(V)}$. \square

REMARK 47. In the case of codimension 1 foliations on manifolds, if there exists a transverse invariant measure Λ of the Lebesgue class, we can change the diameter for the measure in the computation of the secondary dynamical category. In higher codimension $\text{Cat}_{\text{dyn}}(\mathcal{F}, d) = 0 \Rightarrow \text{Cat}(\mathcal{F}, \Lambda) = 0$, but the reverse is not true in general.

EXAMPLE 12.10. The minimal Kröner foliation \mathcal{F}_n on the $(n+1)$ -torus given by minimal hyperplanes satisfies $\text{Cat}_{\text{II}}(\mathcal{F}_n, \Lambda) = [1^n, 2^n, 3^n, \dots]$, where Λ is a Lebesgue transverse invariant measure given in a complete transversal (the usual Lebesgue measure in the transverse circle). Thus, by the previous remark, $\text{Cat}_{\text{II}}(\mathcal{F}, d) = [1^n, 2^n, 3^n, \dots]$.

PROPOSITION 12.11. $\text{Cat}_{\text{dyn,II}}(\mathcal{F} \sqcup \mathcal{G}, d'') = \left\lceil \frac{1}{\text{Cat}(\mathcal{F}, d, n) + \text{Cat}(\mathcal{G}, d', n)} \right\rceil$, where $\mathcal{F} \sqcup \mathcal{G}$ is the disjoint union of the foliations (\mathcal{F}, d) and (\mathcal{G}, d') and the metric d'' is a metric on $\mathcal{F} \sqcup \mathcal{G}$ compatible with d and d' .

Proposition 12.11 is elementary, but it has an interesting consequence. We can identify large secondary categories with rich transverse dynamics and small secondary categories with simple dynamics. Proposition 12.11 shows that simple dynamics may hide interesting dynamics. So we must study also the secondary category of the minimal sets or ergodic components, if possible, in order to obtain more dynamical information.

3. Pseudogroup invariance

We have seen that the nullity or the positivity of the dynamical category is an invariant of the holonomy pseudogroups (Propositions 11.10 and 11.13). Since the secondary category is defined in the case of null category, we ask about its invariance in the same sense. This will be the aim of this section. We shall use the definitions and notations of Section 4.

DEFINITION 12.12 (Secondary dynamical category of pseudogroups). Let Γ be a pseudogroup of local Lipschitz homeomorphisms of a locally compact, second countable metric space T of finite dimension, and let S be a symmetric set of generators. Then define

$$\text{Cat}(\Gamma, d, S, n) = \inf_{\mathcal{U}} \sum_{U \in \mathcal{U}} \text{diam} \left(\bigcup_i h_i^U(U_i) \right)$$

where \mathcal{U} runs in the family of open coverings of T and (h_i^U) are deformations of length $\leq n$ of each $U \in \mathcal{U}$. The secondary category of pseudogroups is defined as the growth type of the sequence $\frac{1}{\text{Cat}(\Gamma, d, S, n)}$: that is,

$$\text{Cat}_{\text{dyn,II}}(\Gamma, d, S) = \left\lceil \frac{1}{\text{Cat}(\Gamma, d, S, n)} \right\rceil,$$

PROPOSITION 12.13. Suppose Γ is a recurrent compactly generated pseudogroup, let U, \mathcal{H} and $S, g \in S$ and $\bar{g} \in \Gamma$ like in Definition 10.18, and let $V \subset U$ and E be like in Definition 10.19. Then $\text{Cat}(\Gamma, d, E) = \text{Cat}(\mathcal{H}, d, S)$.

PROOF. Since E is recurrent and by using the dimensional trick of the proof of the Proposition 11.13, we can suppose that we have a covering $\{U_1, \dots, U_{\dim T+1}\}$ of T and deformations $h_i : U_i \rightarrow V$ with $\text{length}_E(h_i) \leq R$. Since U is relatively compact and any $g \in S$ extends to a map $\bar{g} \in \Gamma$ defined in a neighborhood of \bar{U} , we get that any map $g \in S$ can be locally decomposed into a finite number of restrictions in E_K for K large enough. Also, by taking the components of h_i small enough, any deformation of

length $\leq n$ in S induces a deformation of length $\leq Kn$ in E for K large enough. Hence

$$\text{Cat}(\Gamma, d, Kn + R, E) \leq (\dim T + 1) \cdot \text{Cat}(\mathcal{H}, d, S, n),$$

obtaining $\text{Cat}_{\text{II}}(\mathcal{H}, d, S) \leq \text{Cat}_{\text{II}}(\Gamma, d, E)$. The reverse inequality is easier by using the recurrence of E . There exists a constant K' such that any composition of a map of E restricted to U and a returning map is of length $\leq K'$ (relative to the symmetric set of generators S), and hence $\text{Cat}(\mathcal{H}, d, K'n, S) \leq C \text{Cat}(\Gamma, d, n + R, E)$, where C is an upper bound for the Lipschitz distortions of the returning maps (they have length lower than R , so they are finite). \square

COROLLARY 12.14. *The secondary dynamical category of recurrent compactly generated pseudogroups is independent of the choice of the recurrent set of generators.*

PROOF. The recurrency allows to restrict our study to relatively compact sets and systems of compact generation, for which the independence is clear. \square

REMARK 48. According to Proposition 12.14, we can remove the recurrent set of generators from the notation of the secondary category of pseudogroups, using simply $\text{Cat}_{\text{dyn,II}}(\Gamma)$.

PROPOSITION 12.15. *For recurrent compactly generated pseudogroups, $\text{Cat}_{\text{dyn,II}}(\Gamma)$ is invariant by uniformly bi-Lipschitz étalé equivalences (see Definition 11.9).*

PROOF. Let Φ be a uniformly bi-Lipschitz étalé morphism between (Γ, T, d) and (Γ', T', d') . The pseudogroup Γ_{Φ} in $T \sqcup T'$, generated by Φ , Γ and Γ' is recurrent and compactly generated. The diameters can be related by the uniform Lipschitz distortion of the maps in Φ and Φ^{-1} . \square

REMARK 49. The holonomy pseudogroup given by a regular foliated atlas of a lamination in a compact space is recurrent and compactly generated.

PROPOSITION 12.16. *Let (X, \mathcal{F}, d) be a lamination on a compact metric space of finite dimension such that $\text{Cat}_{\text{dyn}}(\mathcal{F}, d) = 0$. Let Γ be the holonomy pseudogroup of \mathcal{F} on a complete transversal T . Then $\text{Cat}_{\text{dyn,II}}(\mathcal{F}) = \text{Cat}_{\text{dyn,II}}(\Gamma)$.*

PROOF. Let $\mathcal{U} = \{U_1, \dots, U_M\}$ be a regular foliated atlas. The inequality $\text{Cat}_{\text{dyn,II}}(\mathcal{F}) \leq \text{Cat}_{\text{dyn,II}}(\Gamma)$ is obvious since

$$\text{Cat}(\Gamma, d, S^{\mathcal{U}}, n) \leq \text{Cat}(\mathcal{F}, d, \mathcal{U}, n, T).$$

By using the same dimensional trick of the proof of Proposition 11.13, we can show that $\text{Cat}(\mathcal{F}, d, \mathcal{U}, n) \leq M(m+1) \text{Cat}(\Gamma, \Lambda, S^{\mathcal{U}}, n)$, where m is the dimension of X . Let $\{V_i\}_{i \in \mathbb{N}}$ be an open covering of a complete transversal T associated to the atlas \mathcal{U} . Let $\text{sat}_i(B)$ be the saturation of a set B relative to the chart $U_i \in \mathcal{U}$. Of course, $\{\text{sat}_j(V_i \cap T_j)\}_{i,j}$ is an open covering of X . Now let $D_1, \dots, D_{\dim X + 1}$ be an open covering such that $D_i = \bigsqcup_j D_{ij}$, where the sets D_{ij} are open and mutually disjoint, and the whole collection $\{D_{ij}\}_{i,j}$ is a refinement of $\{\text{sat}_j(V_i \cap T_j)\}_{i,j}$. Let $(k, l) = I(D_{ij})$ be the first indexes

(a, b) relative to the lexicographic order such that $D_{ij} \subset \text{sat}_a(V_b \cap T_a)$, and let W_{ijk} be the union of sets D_{il} such that $I(D_{il}) = (j, k)$. Each set W_{ijk} contracts to V_i by a deformation of length 0 (the contraction of a chart to its transversal), and therefore any deformation h_i of length n of V_i induces a tangential deformation H of each W_{ijk} of length n such that $H(W_{ijk} \times \{1\}) \subset h(V_i)$. Hence

$$\sum_{ijk} \text{diam}(H(W_{ijk} \times \{1\})) \leq M(m+1) \sum_i \text{diam}(h_i(V_i)) .$$

□

COROLLARY 12.17. *The secondary dynamical category is a transverse invariant of foliations.*

4. Cohomological upper bound

In this chapter, we use the same notation and background as in Chapter 10, Section 5.

DEFINITION 12.18. The sets of the form $\text{Pen}_{\mathcal{F}}(T, \mathcal{U}, n)$, for a transversal T with $\text{diam}(T) < \varepsilon$, are called $(n, \varepsilon, \mathcal{U}, d)$ -sets.

DEFINITION 12.19. A differentiable leafwise form ω is called an $(n, \varepsilon, \mathcal{U}, d)$ -form if $\text{supp}(\omega)$ is contained in an $(n, \varepsilon, \mathcal{U}, d)$ -set.

DEFINITION 12.20. Define

$$\text{cuplength}(\mathcal{F}, d, n, \mathcal{U}) = \sup_{\xi} \sum_{\varepsilon \in \xi} \varepsilon ,$$

where ξ runs in the family of finite subsets of \mathbb{R}^+ such that, for each $\varepsilon \in \xi$, there is some leafwise closed form α_{ε} of positive degree so that, for all $(n, \varepsilon, \mathcal{U}, d)$ -form β_{ε} , we have $\bigwedge_{\varepsilon \in \xi} (\alpha_{\varepsilon} - d_{\mathcal{F}} \beta_{\varepsilon}) \neq 0$ (even though the exterior product is not commutative, it is commutative up to sign, and therefore this notation makes sense).

REMARK 50. It is possible that $\xi = \emptyset$; in this case, we set

$$\text{cuplength}(\mathcal{F}, d, n, \mathcal{U}) = 0 .$$

PROPOSITION 12.21. $\text{cuplength}(\mathcal{F}, d, n, \mathcal{U}) \leq \text{Cat}(\mathcal{F}, d, n, \mathcal{U})$.

PROOF. Let $\{U_1, \dots, U_K\}$ be an open covering of M , and let F^i be a tangential deformation with length $\leq n$ of each U_i such that

$$\sum_{i=1}^K \text{diam}(F^i(U_i \times \{1\})) < \text{Cat}(\mathcal{F}, d, n, \mathcal{U}) + \delta$$

for some $\delta > 0$. Since $\text{Cat}(\mathcal{F}, d) = 0$, we can suppose, for n large enough, that all sets U_i are tangentially categorical and the maps F^i are tangential contractions. Hence the leafwise forms supported in each U_i are $(n, \varepsilon_i, \mathcal{U}, d)$ -forms, where $\varepsilon_i = \text{diam}(F^i(U_i \times \{1\}))$. Since $\{U_1, \dots, U_K\}$ is a covering of M , a classical cohomological argument shows that, for any leafwise form α_i , $i \in \{1, \dots, K\}$, there exists an $(n, \varepsilon_i, \mathcal{U}, d)$ -form β_i such that

$$\bigwedge_{i=1}^K (\alpha_i - d\beta_i) = 0 . \quad \square$$

DEFINITION 12.22. The *secondary dynamical cuplength* is

$$\text{cuplength}_{\text{II}}(\mathcal{F}, d, \mathcal{U}) = \left\lceil \frac{1}{\text{cuplength}(\mathcal{F}, d, n, \mathcal{U})} \right\rceil.$$

PROPOSITION 12.23. *The secondary dynamical cuplength does not depend on the choice of the regular foliated atlas.*

PROOF. The proof is quite similar to the proof of the independence of the choice of the atlas for the definition of the secondary category. Let \mathcal{U} and \mathcal{V} be regular foliated atlases. Let K be the minimum length such that any pair of charts of \mathcal{U} with non-empty intersection is covered by chains of charts in \mathcal{V} of length $\leq K$. Then any $(n, \varepsilon, \mathcal{U}, d)$ -form is a $(Kn, \varepsilon, \mathcal{V}, d)$ -form, and therefore

$$\text{cuplength}(\mathcal{F}, d, Kn, \mathcal{V}) \leq \text{cuplength}(\mathcal{F}, d, n, \mathcal{U}),$$

yielding $\text{cuplength}_{\text{II}}(\mathcal{F}, d, \mathcal{U}) \leq \text{cuplength}_{\text{II}}(\mathcal{F}, d, \mathcal{V})$. \square

COROLLARY 12.24. $\text{Cat}_{\text{dyn,II}}(\mathcal{F}) \leq \text{cuplength}_{\text{II}}(\mathcal{F}, d)$.

5. Semicontinuity of the dynamical and secondary dynamical category

Recall that we have adapted for the Λ -category and the secondary Λ -category a result of W. Singhof and E. Vogt [33], which asserts that the tangential category is an upper semicontinuous map defined on the space of C^2 foliations over a C^∞ closed manifold M . Now we show that it also has a version for the secondary dynamical category.

In order to adapt this result for our dynamical invariants, we work in the same basic set-up and with the same basic notation as in Section 8: For a C^∞ compact manifold M , we consider the space $\text{Fol}_p^1(M)$ of C^2 foliations of dimension p on M . We also consider a fixed metric d on M . Now we recall Proposition 8.2, and we shall show how the semicontinuity of the dynamical invariants is a corollary of this proposition.

COROLLARY 12.25 (Upper semicontinuity of dynamical category). *Let $(\mathcal{F}_n)_{n \in \mathbb{N}}$ be a sequence in $\text{Fol}_p^1(M)$ converging to \mathcal{F} . Then, for any $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that $\text{Cat}_{\text{dyn}}(\mathcal{F}_n, d) \leq \text{Cat}_{\text{dyn}}(\mathcal{F}, d) + \varepsilon$ for all $n \geq N$.*

PROOF. By Proposition 8.2, tangential deformations in \mathcal{F} can be approximated by tangential deformations in \mathcal{F}_n . Then the result is given by the continuity of the metric when the approximations are close enough, which can be achieved for n large enough. \square

COROLLARY 12.26. *Let $(\mathcal{F}_n)_{n \in \mathbb{N}}$ be a sequence in $\text{Fol}_p^1(M)$ converging to \mathcal{F} such that $\text{Cat}(\mathcal{F}_n, d) = \text{Cat}(\mathcal{F}, d) = 0$ for all n . Then there exists some $N \in \mathbb{N}$ such that $\text{Cat}_{\text{dyn,II}}(\mathcal{F}_n, \Lambda_n) \geq \text{Cat}_{\text{dyn,II}}(\mathcal{F}, \Lambda)$ for all $n \geq N$.*

PROOF. We can use simultaneous local parametrizations (see Definition 8.4) in order to choose regular atlases for the foliations \mathcal{F}_n . Now, by using the same argument as in Theorem 10.34 and by Proposition 8.2, for all $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that

$$\text{Cat}(\mathcal{F}_n, d, \mathcal{U}_n, k) \leq \text{Cat}(\mathcal{F}, d, \mathcal{U}, k) + \varepsilon$$

for all $n \geq N$, where \mathcal{U}_n and \mathcal{U} are regular atlases given by simultaneous local parametrizations associated to \mathcal{F}_n and \mathcal{F} , respectively. Therefore $\text{Cat}_{\text{dyn}, \Pi}(\mathcal{F}, d) \leq \text{Cat}_{\text{dyn}, \Pi}(\mathcal{F}_n, d)$. \square

REMARK 51. These corollaries are also true if we allow a continuous (or even upper semicontinuous) variation of the metric d .

6. Critical points

Let (M, \mathcal{F}, d) be a C^∞ lamination on a compact metric space. Suppose that $\text{Cat}_{\text{dyn}}(\mathcal{F}) = 0$. Let \mathcal{U} be a finite regular foliated atlas. For each $f \in C^2(M)$ with isolated critical points on the leaves, let $\Phi(f)$ denote the leafwise flow associated to the leafwise gradient of f . Define the critical sets relative to f , $C_i(f)$, like in Chapter 9.

THEOREM 12.27. *Let (M, \mathcal{F}, d) be a Hilbert lamination endowed with a Riemannian metric on leaves. Let f be an ω -PS function satisfying the hypothesis of Theorem 9.9. Then*

$$\text{Cat}(\mathcal{F}, d) \leq \sum_i \text{diam}(C_i(f)) \leq \text{diam}(\text{Crit}_{\mathcal{F}}(f)) .$$

PROOF. Use the same deformations as in the proof of Theorem 9.9. Then we obtain a covering by open sets that contract tangentially to open neighborhoods of the critical sets. These neighborhoods can be chosen with diameter arbitrarily close to the diameter of the corresponding critical set. \square

Finally, we relate the secondary category to the leafwise critical point sets of functions. The secondary category was defined for compact foliated spaces; thus we restrict this study to the case of usual foliations on compact spaces, without considering Hilbert foliations. However it could be generalized to non-compact foliated spaces, and even Hilbert foliations, but then it would depend strongly on the choice of the regular atlas. Recall here the definition of (n, \mathcal{U}) -bounded map (Definition 10.36).

PROPOSITION 12.28. *Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of ω -PS functions on M and suppose that each f_n is (n, \mathcal{U}) -bounded. Then*

$$\left[\frac{1}{\text{diam}(\text{Crit}_{\mathcal{F}}(f_n))} \right] \leq \left[\frac{1}{\sum_i \text{diam}(C_i(f))} \right] \leq \text{Cat}_{\text{dyn}, \Pi}(\mathcal{F}, d) .$$

PROOF. The deformations in the proof of Theorem 9.9 follow the flow-lines of the leafwise gradient flow. Hence, since f_n is (n, \mathcal{U}) -bounded,

$$\text{Cat}(\mathcal{F}, d, n, \mathcal{U}) \leq \sum_i \text{diam}(C_i(f_n)) \leq \text{diam}(\text{Crit}_{\mathcal{F}}(f_n))$$

for any n . \square

Concluding remarks

We have introduced several new versions of the tangential LS category for laminations, computed them in examples, extended the classical results about LS category to this set-up, and proved new theorems that are genuine of laminations. Hopefully, they will play a role in the study of foliations. Particularly, there are two promising lines of research to continue this study:

- (1) The relation between the tangential category and critical points can be possibly used to prove foliation versions of variational problems where the LS category was used, like the relation between the number of geodesics and the topology; in the foliation set-up, one should get a relation between the “number” of geodesics on the leaves and transverse invariants of foliations.
- (2) Certain dual secondary tangential categories can be also defined as follows. Let $\epsilon(L)$ be the value of the expression used to define the primary tangential category with tangential deformations of length $\leq L$. Dually, let $L(\epsilon)$ be the length of deformations needed to obtain an expression $< \epsilon$ used to define the primary tangential category. Now, instead of considering the growth type of $1/\epsilon(L)$ as $L \rightarrow \infty$ (in the secondary category), consider the growth type of $L(\epsilon)$ as $\epsilon \rightarrow 0$. This second point of view produces a dual secondary (Λ - or dynamical) category. This definition is very similar to the definition of entropy, which is a very important dynamical invariant. Thus, besides of trying to generalize the type of results of the thesis to this new invariant, we could try to relate it to the entropy. Such a relation would establish a very relevant connection between different aspects of foliations: on the one hand, this category would be related with critical points and cup-length, and, on the other hand, entropy is related with secondary characteristic classes, resilient leaves and existence of transverse invariant measures.

Roughly speaking, the following ideas from the thesis could be kept in mind. The tangential deformations of open sets is a way of representing the dynamics of the foliation (given by sliding sets along the leaves). Then the primary categories indicate how small are the transversals where we can “hide” the topology of the leaves with the dynamics of the foliation. They vanish when we can “hide” the topology of the leaves in “small” transversals. In this case, the secondary categories indicate how fast can we “hide” the leaf topology in “small” transversals. So they surely contain important information about the relation between the leaf topology and the dynamics

of a foliation. This thesis can be considered as a step towards revealing and using that content.

Resumen de la Tesis

Categoría LS, espacios foliados y medida transversa invariante

Introducción

La categoría de Lusternik-Schnirelmann es un invariante homotópico que mide el número mínimo de abiertos tangencialmente categóricos que cubren un espacio topológico dado. Fue introducido en 1935 en el contexto del cálculo de variaciones [28]. La categoría LS ha sido adaptada a foliaciones de varias maneras a lo largo de los últimos años. Entre esas adaptaciones, destacan por su generalidad la categoría tangente y la categoría transversa [7, 8]. Muchas propiedades de categoría LS clásica fueron adaptadas a estos dos contextos siendo la categoría transversa la más desarrollada. Nuestro enfoque es el de la categoría tangente, donde trabajamos con abiertos y deformaciones que conservan las hojas de la foliación. Los abiertos tangencialmente contráctiles son la pieza clave en la definición de la categoría tangente. Este tipo de abiertos se contraen tangencialmente a una transversal de la foliación de manera que se puede tomar la medida de esta transversal si nuestra foliación dispone de una medida transversa invariante. Surge así el concepto de Λ -categoría LS de foliaciones con una medida transversa invariante Λ .

Muchas propiedades de la categoría tangente se adaptan a la Λ -categoría, como pueden ser la invarianza de homotopía tangente, la semicontinuidad en un espacio apropiado de foliaciones, o la existencia de una cota superior dependiente de la dimensión de la foliación [33]. Resulta que la Λ -categoría se anula con mucha frecuencia; por ejemplo, toda foliación minimal (no trivial) tiene Λ -categoría nula. No obstante, el hecho de ser positiva o nula convierte a la Λ -categoría en un invariante transverso.

Intentando extraer más información en el caso de categoría nula, definimos un invariante secundario como el tipo de decrecimiento a cero de la expresión que define Λ -categoría. Este tipo de decrecimiento resulta ser un invariante transverso. También las propiedades clásicas de la categoría LS se adaptan a este invariante secundario. Sin embargo el resultado más sorprendente es la relación directa que existe entre la categoría secundaria y el crecimiento del pseudogrupo, siendo la primera una cota inferior para la segunda y dándose la igualdad en el caso de suspensiones por grupos de Rohlin. Una consecuencia de esto es que la Λ -categoría secundaria extrae información no trivial en el caso de Λ -categoría nula.

En la última parte de la tesis, se usa el diámetro respecto de una métrica ambiente para adaptar la definición de Λ -categoría para los casos de foliaciones sin medida transversa invariante, ya que la existencia de tales medidas es una condición muy restrictiva. Surge así la categoría dinámica que hereda muchas de las propiedades de la Λ -categoría. En especial, también tenemos una relación con el crecimiento del pseudogrupo, si bien en este caso es más compleja.

Finalmente, observamos que el principal resultado de la categoría LS clásica es que sirve como cota inferior para el número de puntos críticos de funciones diferenciables en variedades de Hilbert [31]. A lo largo de la tesis este resultado será adaptado para todas las categorías definidas y también para la categoría tangente clásica, siendo en este contexto la primera vez que se hace. El punto clave ahora será usar funciones diferenciables en las hojas y tomar conjuntos críticos formados por los puntos críticos que se generan hoja a hoja; para ello se asume que los puntos críticos son aislados respecto de la topología de las hojas.

La tesis se divide en cuatro partes diferenciadas. La primera trabaja los contextos de categoría tangente y Λ -categoría en el contexto de foliaciones medibles, las cuales son foliaciones en las que la topología ambiente se cambia por una estructura medible estándar [4]. Remarcamos que este estudio dio como fruto la introducción de la cohomología singular medible. La longitud del producto “cup” de esta cohomología dará una cota inferior para la categoría tangente. En la segunda parte, directamente inspirada en la tesina del autor, se trabajan los conceptos de Λ -categoría en el contexto de laminaciones topológicas. La tercera se dedica a introducir la Λ -categoría secundaria y probar sus principales resultados. En la última parte se hace lo propio con la categoría dinámica y la categoría dinámica secundaria.

Resumen de la parte I

En una foliación medible (X, \mathcal{F}) , un abierto medible $U \subset X$ se dice que es tangencialmente categórico si existe una deformación $H : U \times \mathbb{R} \rightarrow X$ continua en las hojas y medible ambientalmente tal que $H(x, 0) = x$ y $H(L_U \times \{1\}) = *$ para toda hoja L_U de \mathcal{F}_U , donde \mathcal{F}_U es la restricción de \mathcal{F} a U . La categoría tangente medible es el menor número de abiertos tangencialmente categóricos que recubren a una laminación medible.

Cuando tenemos una medida transversa invariante Λ , podemos hablar de una extensión coherente $\tilde{\Lambda}$ al espacio ambiente. Dicha extensión se define mediante la integración de la medida de contar en las hojas respecto de la medida transversa invariante. Para un abierto medible $U \subset X$, se define $\tau_\Lambda(U) = \inf_H \tilde{\Lambda}(H(U \times \{1\}))$, donde H recorre todas las deformaciones medibles de U . La Λ -categoría se define como

$$\text{Cat}(\mathcal{F}, \Lambda) = \inf_{\mathcal{U}} \sum_{U \in \mathcal{U}} \tau_\Lambda(U),$$

donde \mathcal{U} recorre los recubrimientos numerables de X por abiertos medibles.

La categoría tangente medible es invariante por equivalencias de homotopía medible. La Λ -categoría es también un invariante del mismo tipo de

equivalencias, siempre y cuando las medidas transversas invariantes sean respetadas por dicha equivalencia.

En foliaciones por hojas compactas, la medida invariante Λ pasa al espacio de hojas X/\mathcal{F} como una medida usual $\Lambda_{\mathcal{F}}$. Se comprueba que su Λ -categoría se puede calcular como una integración en el espacio de hojas de la función categoría LS clásica en las hojas:

$$\text{Cat}_{\text{meas}}(\mathcal{F}, \Lambda) = \int_{X/\mathcal{F}} \text{Cat}(L) d\Lambda_{\mathcal{F}}(L) .$$

En el caso de tener una estructura diferenciable en las hojas, se tiene que $\text{Cat}_{\text{meas}}(\mathcal{F}) \leq \dim \mathcal{F} + 1$. Para la Λ -categoría tenemos un resultado similar: si T es una transversal medible completa, entonces $\text{Cat}_{\text{meas}}(\mathcal{F}, \Lambda) \leq (\dim \mathcal{F} + 1) \cdot \Lambda(T)$.

Se define la cohomología singular medible a partir del subcomplejo de cocadenas medibles. Las cocadenas medibles son aquellas que inducen aplicaciones medibles al actuar sobre un prisma medible. Más precisamente, si $\Delta \times T$ es un prisma medible, es decir, un producto de un símplice canónico con un espacio estándar, y $\sigma : \Delta \times T \rightarrow \mathcal{F}$ es una aplicación medible y continua, entonces la aplicación $\omega(\sigma|_{\Delta \times \{-\}}) : T \rightarrow \Gamma$ es medible, donde Γ es un anillo medible de coeficientes.

La teoría de cohomología medible resulta satisfactoria. Un capítulo se dedica a desarrollarla, y se obtiene que el orden de nilpotencia de cohomología singular medible es una cota inferior para la categoría tangente. Como aplicación de este resultado, se calcula la categoría tangente de las foliaciones minimales en los toros n -dimensionales dadas por $n-1$ -hiperplanos, \mathcal{F}_n , obteniendo $\text{Cat}(\mathcal{F}_n) = n$.

Finalmente, se prueba el primer resultado de puntos críticos para estas dos categorías. El resultado que adaptamos a lo largo de toda la tesis es el siguiente debido a J. Schwartz [31]:

TEOREMA 1. *Sea M una variedad de Hilbert C^2 y sea $f : M \rightarrow \mathbb{R}$ una función C^2 de Palais-Smale y acotada inferiormente. Entonces*

$$\text{Cat}(M) \leq \#\{\text{puntos críticos de } f\} .$$

Las funciones de Palais-Smale cumplen la siguiente condición: si (x_n) es una sucesión tal que $(\nabla f(x_n))$ converge a cero y $(f(x_n))$ es acotada, entonces existe una subsucesión (x_{n_k}) convergente. En el caso de foliaciones medibles, tomaremos funciones diferenciables en las hojas cumpliendo una condición de Palais-Smale adaptada que denominamos condición ω -Palais-Smale. Esta condición tiene sentido incluso en el caso de variedades y es más general que la condición de Palais-Smale usual, de manera que nuestro teorema adaptado incluso introduce una mejora en el teorema clásico de Schwartz.

El papel de los puntos críticos es ahora el de los conjuntos críticos. El primero de ellos es el conjunto de mínimos relativos en las hojas, y se continúan definiendo recursivamente usando el flujo gradiente. Se dice que dos puntos críticos, p y q , están relacionados, $p < q$, si existe una curva integral del flujo gradiente tal que p y q están, respectivamente, en el α - y ω -límites de dicha curva. El segundo conjunto crítico se define como aquellos

puntos críticos que solo se relacionan inferiormente con puntos del primer conjunto crítico y así sucesivamente.

Obtenemos los dos siguientes teoremas de puntos críticos:

TEOREMA 2. *Sea (X, \mathcal{F}) una foliación de Hilbert medible y $f : X \rightarrow \mathbb{R}$ una función medible, C^2 en las hojas y ω -Palais-Smale. Entonces*

$$\text{Cat}(\mathcal{F}) \leq \#\{\text{conjuntos críticos de } f\}.$$

TEOREMA 3. *Sea $(X, \mathcal{F}, \Lambda)$ una foliación de Hilbert medible con una medida transversa invariante regular y $f : X \rightarrow \mathbb{R}$ una función medible, C^2 en las hojas y ω -Palais-Smale. Entonces*

$$\text{Cat}(\mathcal{F}, \Lambda) \leq \Lambda(\{\text{puntos críticos de } f\}).$$

Resumen de la parte II

En esta segunda parte, estudiamos la Λ -categoría en laminaciones topológicas. La definición es análoga a su versión medible, solo que ahora tomaremos conjuntos abiertos ambientalmente y deformaciones ambientalmente continuas.

Lo primero que observamos es que las versiones medibles son cotas inferiores para sus versiones topológicas; es decir, $\text{Cat}_{\text{meas}}(\mathcal{F}, \Lambda) \leq \text{Cat}_{\text{top}}(\mathcal{F})$ y $\text{Cat}_{\text{meas}}(\mathcal{F}, \Lambda) \leq \text{Cat}_{\text{top}}(\mathcal{F}, \Lambda)$.

Las versiones topológicas son invariantes de equivalencias por homotopía tangente. Más aun, obtenemos aquí un resultado adicional, el carácter positivo o nulo de la Λ -categoría es un invariante transverso.

También obtenemos un resultado que relaciona, en alguna medida, la categoría tangente y la Λ -categoría cuando la medida transversa invariante es positiva en abiertos no vacíos. Se obtiene

$$\tau_{\Lambda}(U) < \infty \implies U \text{ es tangencialmente categórico.}$$

Se calcula la Λ -categoría topológica en foliaciones compactas-Hausdorff, obteniendo la misma expresión que tenemos para la categoría medible en foliaciones compactas. Se presenta un ejemplo donde las dos Λ -categorías medible y topológica difieren, si bien, la razón fundamental es que la medida invariante escogida no es finita en conjuntos compactos.

Adaptamos el resultado de W. Singhof y E. Vogt [33], que establece que la categoría tangente es una función superiormente continua en el espacio de foliaciones C^2 sobre una variedad compacta. En nuestro caso, la Λ -categoría es una función superiormente semicontinua en el espacio de foliaciones C^2 con medida transversa invariante sobre una variedad compacta. Usaremos la topología débil de las medidas para definir la topología de este espacio.

Para foliaciones C^2 en una variedad compacta y para una transversal completa T , se tiene $\text{Cat}(\mathcal{F}, \Lambda) \leq (\dim \mathcal{F} + 1) \cdot \Lambda(T)$. En general, sin las condiciones de diferenciabilidad ni compacidad, se tiene $\text{Cat}(\mathcal{F}, \Lambda) \leq (\dim X + 1) \cdot \Lambda(T)$, donde X es el espacio ambiente. Como corolario obtenemos que las foliaciones minimales o ergódicas no triviales tienen categoría nula.

Respecto a la relación con puntos críticos, los resultados obtenidos son análogos a los que obtuvimos en el caso medible. Lo único nuevo que hay que indicar ahora es que debemos exigir que los conjuntos críticos sean

cerrados. Esta condición es restrictiva, sin embargo creemos que el resultado es igualmente cierto sin ella. Para categoría tangente, este resultado es el primero en su naturaleza.

Resumen de la parte III

Ya vimos que la Λ -categoría tiende a anularse a poco que la foliación se complique, lo cual es una mala noticia. Para intentar sacar información en estos casos, definimos la Λ -categoría secundaria. Para ello, definimos primero la (Λ, n) -categoría del mismo modo que la Λ -categoría, pero tomando deformaciones que en cada hoja tienen una longitud menor o igual que n , usando cadenas de cartas de un atlas foliado regular para medir esa longitud. La sucesión de (Λ, n) -categorías converge a la Λ -categoría cuando $n \rightarrow \infty$. Si ésta se anula, los inversos forma una sucesión divergente, cuyo tipo de crecimiento, $\text{Cat}_{\Pi}(\mathcal{F}, \Lambda)$, se denomina Λ -categoría secundaria. De nuevo, la categoría secundaria es un invariante de equivalencias de homotopía tangente así como un invariante de transverso.

Obtenemos un resultado interesante de estimación: la categoría secundaria está acotada superiormente por el crecimiento del pseudogrupo (relativo a un conjunto de generadores proveniente de los cambios de cartas en un atlas foliado regular). La igualdad se da en suspensiones por grupos de Rohlin.

También disponemos de una cota en cohomología, en este caso se define un (Λ, n) -orden de nilpotencia del anillo de cohomología foliada. Para ello se toman formas cerradas cuyo soporte se encuentre en la penumbra de radio n de una transversal de la foliación. El (Λ, n) -orden de nilpotencia se define como el supremo en la suma de las medidas de transversales para las cuales existen formas cerradas en su penumbra con producto “cup” no trivial en cohomología foliada. La sucesión de (Λ, n) -órdenes de nilpotencia converge a cero y el tipo de crecimiento de sus inversos es una cota superior para la categoría secundaria.

Se tiene también la semicontinuidad para la categoría secundaria. Básicamente es un corolario del resultado análogo que se obtuvo para Λ -categoría. Si bien hay que tener en cuenta que los valores son tipos de crecimiento y no números reales.

Para obtener un resultado de puntos críticos, se define el concepto de (n, ∇) -función. Dichas funciones son aquellas cuyas curvas integrales del flujo gradiente tienen longitud $\leq n$ (de nuevo respecto de un atlas foliado regular). Si tenemos una sucesión (f_n) de (n, ∇) -funciones que son además ω -Palais-Smale, entonces el tipo de crecimiento de los inversos de la sucesión $(\Lambda(\{\text{puntos críticos de } f_n\}))$ es una cota inferior para la Λ -categoría secundaria.

Resumen de la parte IV

Finalmente, definimos la categoría dinámica y la categoría dinámica secundaria. Las definiciones son análogas a las de Λ -categoría y Λ -categoría secundaria, donde cambiamos la medida transversa invariante por el diámetro relativo a una métrica ambiente. De este modo podemos definir un concepto similar al de Λ -categoría para foliaciones sin medida transversa invariante.

La similitud de ambos la podemos encontrar en foliaciones de codimensión 1 con medida transversa invariante de la clase de Lebesgue. En este caso, los dos conceptos son en realidad el mismo.

Las categorías dinámicas también resultan ser invariantes de equivalencias de homotopía tangente Lipschitz. El carácter positivo o nulo de la categoría dinámica es también un invariante transverso y no depende del tipo de uniformidad de la métrica escogida.

La categoría dinámica secundaria será también un invariante transverso, si bien las equivalencias entre pseudogrupos deberán tomarse Lipschitz respecto de las métricas. La categoría dinámica secundaria no va a depender del tipo de quasi-isometría de la métrica.

También disponemos de una cota dimensional. Sea \mathcal{F} una laminación en un espacio métrico compacto (X, d) . Si (T_n) es una colección de transversales cuya unión corta a todas las hojas, entonces

$$\text{Cat}(\mathcal{F}, d) \leq (\dim X + 1) \cdot \sum_n \text{diam}(T_n).$$

De hecho $\text{Cat}(\mathcal{F}, d) \leq (\dim \mathcal{F} + 1) \cdot \sum_n \text{diam}(T_n)$ para el caso de C^2 -foliaciones en una variedad compacta. Como consecuencia, la categoría dinámica se anula con la misma facilidad que la Λ -categoría, lo cual justifica el introducir el correspondiente invariante secundario.

La categoría dinámica secundaria también se relaciona con el crecimiento del pseudogrupo. Es una cota inferior para el producto del crecimiento del pseudogrupo por un factor exponencial. No se puede prescindir de este factor exponencial; no obstante, esto puede ser conseguido para foliaciones cuyo pseudogrupo de holonomía cumpla una condición de Lipschitz uniforme.

La categoría dinámica resulta ser una función semicontinua en el espacio de foliaciones C^2 sobre una variedad compacta. Como corolario directo, también obtenemos la semicontinuidad de la categoría dinámica secundaria. De nuevo, estos dos resultados son consecuencias directas del trabajo de W. Singhof y E. Vogt [33].

También se tiene una cota en cohomología. Para este caso debemos definir el (d, n) -orden de nilpotencia, que es análogo al (Λ, n) -orden de nilpotencia, cambiando la medida invariante por el diámetro relativo a la métrica d .

Finalmente, la relación con puntos críticos viene a ser un corolario del teorema principal para categoría tangente:

TEOREMA 4. *Sea (X, \mathcal{F}) una foliación de Hilbert y $f : X \rightarrow \mathbb{R}$ una función tangencialmente C^2 y ω -Palais-Smale. Entonces*

$$\begin{aligned} \text{Cat}(\mathcal{F}, d) &\leq \sum_n \text{diam}(n\text{-ésimo conjunto crítico de } f) \\ &\leq \text{diam}(\{\text{puntos críticos de } f\}). \end{aligned}$$

Para la categoría secundaria, dada una sucesión (f_n) de (n, ∇) -funciones que son además ω -Palais-Smale, el tipo de crecimiento de los inversos de $(\sum_i \text{diam}(\{i\text{-ésimo conjunto crítico de } f_n\}))$ es una cota inferior para la categoría dinámica secundaria.

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